COMP 471
Fourier Transform
Wednesday 25 Sep 2006

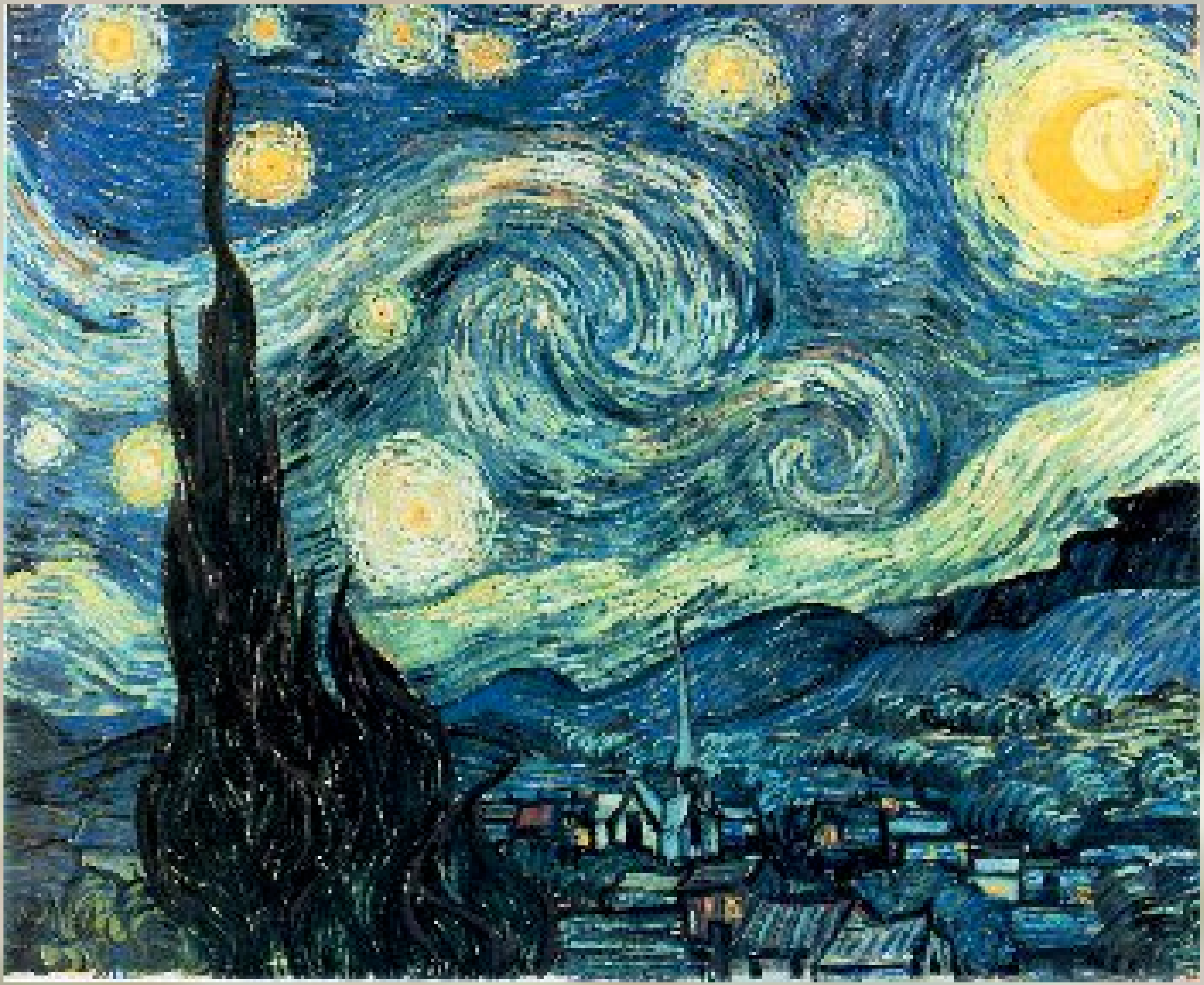
- Sinusoidal Image
- Discrete Fourier Transform
- Meaning of Image Frequencies of DFT

Sinusoidal Images

- We shall make frequent discussion in this module of image frequency content.
- The image having the simplest frequency content is the sinusoidal image.

Sinusoidal Images

- A discrete sine image I has elements
- $I(i, j) = \sin [2\pi (i\, M/u + j\, N/v)]$
- for $0 \leq i \leq N-1,\ 0 \leq j \leq M-1$
- and a discrete cosine image has elements
- $I(i, j) = \cos [2\pi(i\, M/u + j\, N/v)]$
- where u,v are integer frequencies in the i- and j-directions (cycles/image).



- Spatial Frequencies (this is...)

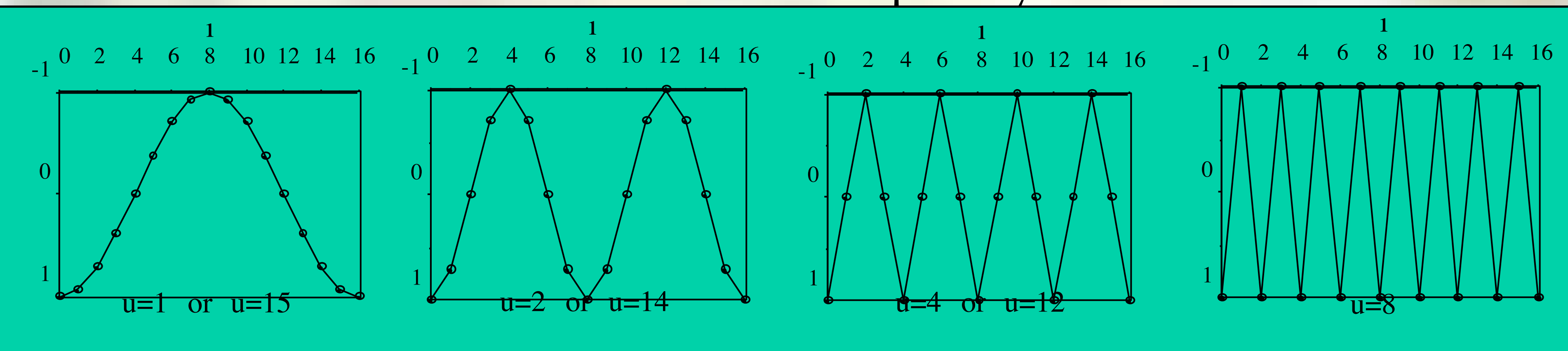
Radial Frequency

- The radial frequency (how fast the image oscillates in its direction of propagation) is
- $$\Omega = \sqrt{u^2 + v^2}$$
- The angle of the wave (relative to i-axis) is

$$\theta = \arctan\left(\frac{v}{u}\right)$$

Digital Sinusoidal Example

- Let N = 16, v = 0: I(i) = cos (2πui/16): a cosine wave oriented in i-direction with frequency u. One row:



- Note that I(i) = cos (2πui/16) = cos [2π(16-u)i/16].
- Thus the highest frequency wave occurs at u = N/2 (N is even here). This will be important later.

Values of Complex Exponential

- The complex exponential

$$W_K^{um} = e^{-2\pi\iota \frac{u}{M} m}$$

- is a frequency representation indexed by exponent ui.
- Minimum physical frequencies: If u = kM, then as a function of m

$$W_M^{kMm} = 1 \; \forall k \in \mathbb{Z}$$

- Maximum physical frequencies: If u = (k+1/2)M, then

$$W_M^{um} = W_M^{(k+\frac{1}{2})Mm} = (-1)^m$$

period 2 function of m (Q. What could this look like as bitmap?)

Complex Exponential Image

- We'll use complex exponential functions to define the Discrete Fourier Transform.
- Define the 2-D complex exponential functions of (u,v):
- $$e^{2\pi\sqrt{-1}(Um+Vn)}$$
- $0 \leq m \leq M\text{-}1,\ 0 \leq n \leq N\text{-}1$
- The complex exponential allows convenient representation and manipulation of frequencies.

Properties of Complex Exponential

- We will use the abbreviation

$$W_K = exp[-\frac{2\pi\iota}{K}]$$

$$\iota = \sqrt{-1}$$

(K = image dimension, M or N).

- Powers of W_K index the frequencies of the component sinusoids.
- This gives functions of (u,v):

$$W^{um}{}_K W^{vn}{}_K = cos[2\pi(\frac{u}{M}m + \frac{v}{N}n)] - \iota sin[2\pi(\frac{u}{M}m + \frac{v}{N}n)]$$

using Euler's identity $e^{\iota\theta} = cos(\theta) - \iota sin(\theta)$

- Will be our *basis* functions for the finite images

Basis Functions

$$W_{mn}[u,v] = W_M^{um} W_N^{vn}$$

These basis functions $W_{mn}[u,v] = W_M^{um} W_N^{vn}$ are orthogonal w/r to this inner product:

$$\sum_{u=0}^{M-1} \sum_{v=0}^{N-1} W_{mn}[u,v] W_{pq}[-u,-v] = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} W_M^{um} W_M^{-up} W_N^{vn} W_N^{-vq}$$

= M N if m=n and p=q,

0 otherwise

Comments

- It's possible to develop frequency domain concepts w/o complex numbers - but the math is much lengthier.
- Using $W_{mn}[u,v] = W_M^{um} W_N^{vn}$ to represent a frequency component oscillating at u (cycles/image) and v (cy/im) in the M- and N-directions simplifies things considerably.
- It is useful to think of W_{mn} as a representation of a direction and frequency of oscillation.

DISCRETE FOURIER TRANSFORM

$$\widetilde{F}[u,v] = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} \boldsymbol{f}[m,n] W_M^{um} W_N^{vn}$$

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= M N if m=n and p=q,

0 otherwise

Inverse Discrete Fourier Transform

$$\boldsymbol{f}[m,n] = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}\widetilde{F}[u,v]W_M^{-um}W_N^{-vn}$$

- Any MxN image **f**[m,n], 0≤m≤M-1, 0≤n≤N-1 is uniquely expressed as the weighted sum of a finite number of complex exponential images.

The weights $\widetilde{F}[u,v]$ are unique. (Why?)

properties of DFT

- If $\tilde{F}$ is the DFT of f
- Linear: DFT[af + bg] = a DFT[f] + b DFT[g]
- Invertible
- Symmetric:
- Periodic => image periodicity

Convolution

- f*g <--> F G ?
- true for infinite images...

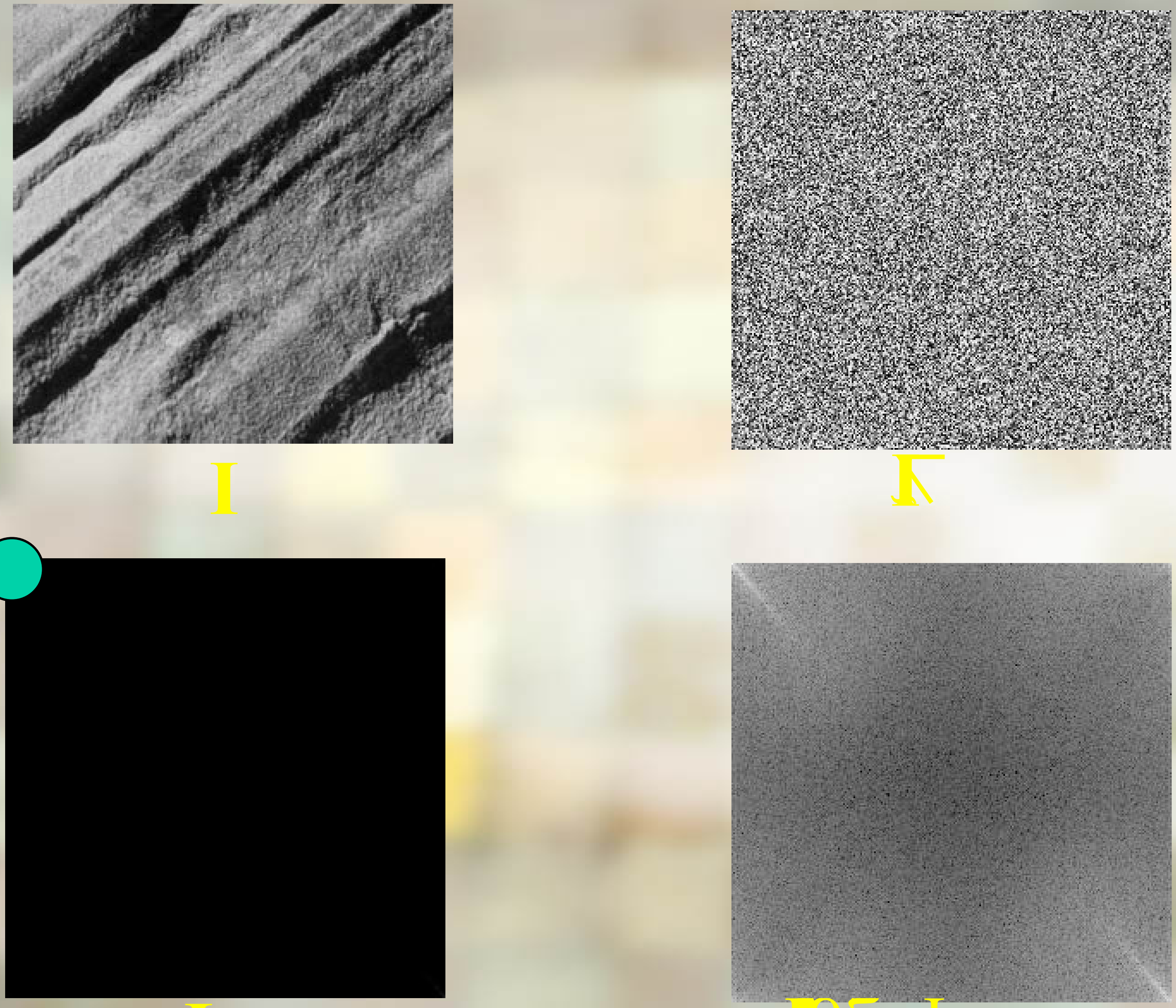
Interpreting the DFT

- The DFT of an image is usually displayed as images of magnitude and of phase.

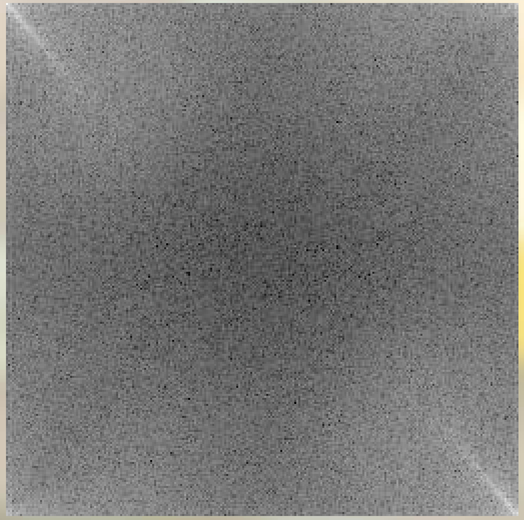
- The magnitude and phase values are given gray-scale values / intensities.

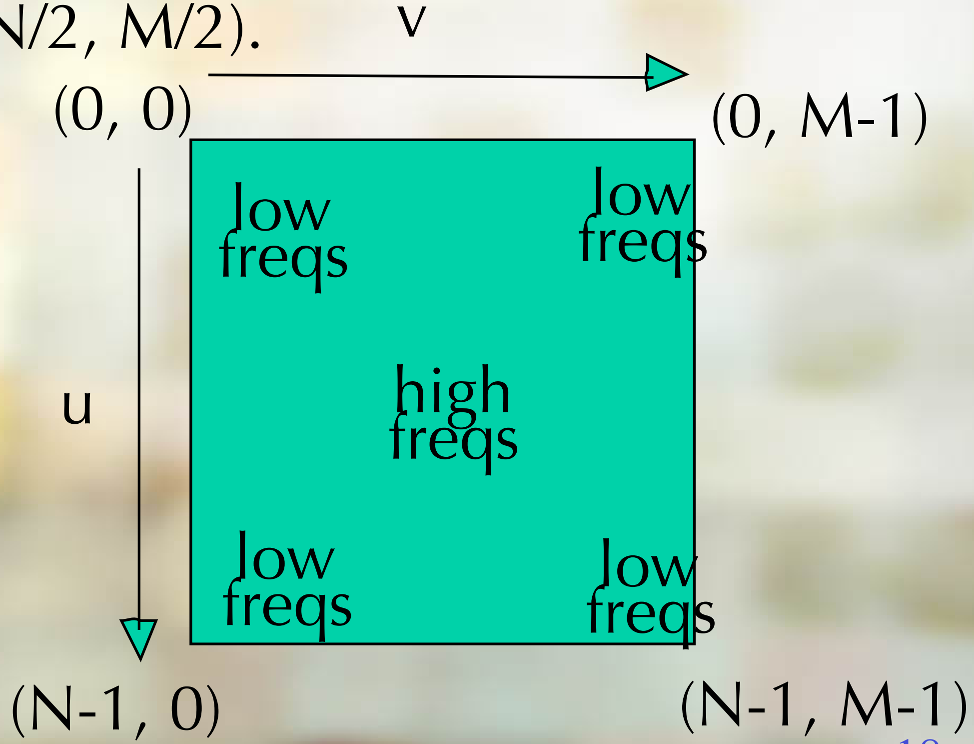
- The phase is usually visually meaningless.

- The magnitude matrix is usually logarithmically transformed (followed by a FSCS) prior to display:



- Note that the coefficients of the highest physical frequencies are located near the center of the DFT matrix: near (u, v) = N/2, M/2).





Periodicity of the DFT

- The DFT matrix is finite (M x N):

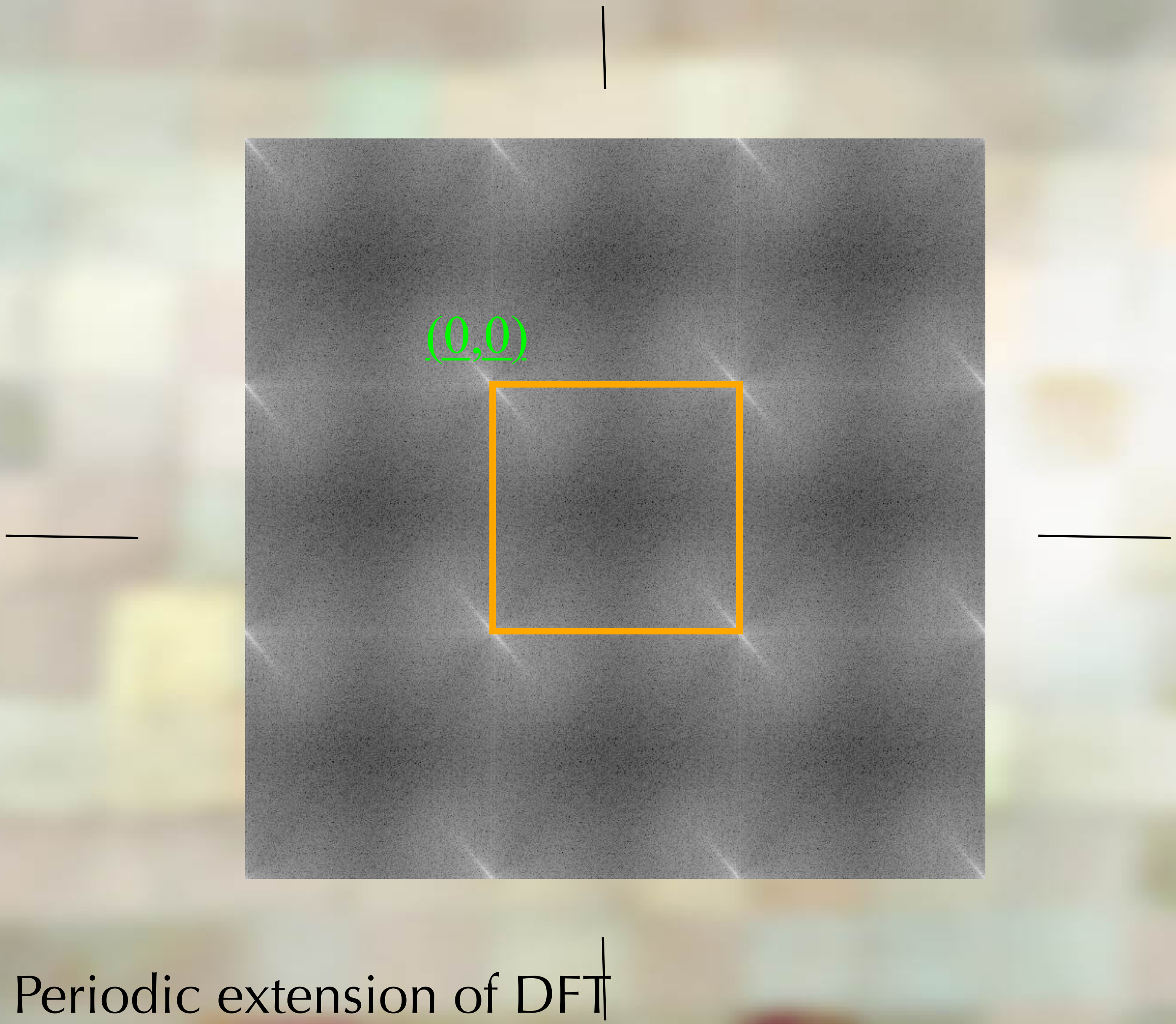
$$\widetilde{F}[u,v] = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \boldsymbol{f}[m,n] W_M^{um} W_N^{vn}$$

$0 \leq u \leq M-1$, $0 \leq v \leq N-1$

- Yet if the indices are allowed to range outside [0,M-1]x[0,N-1], we see the DFT is periodic with periods M and N:

$$\widetilde{F}[u + aM, v + bN] = \widetilde{F}[u,v]$$

for any integers a and b.



- Periodic extension of DFT

Periodic Extension of Image

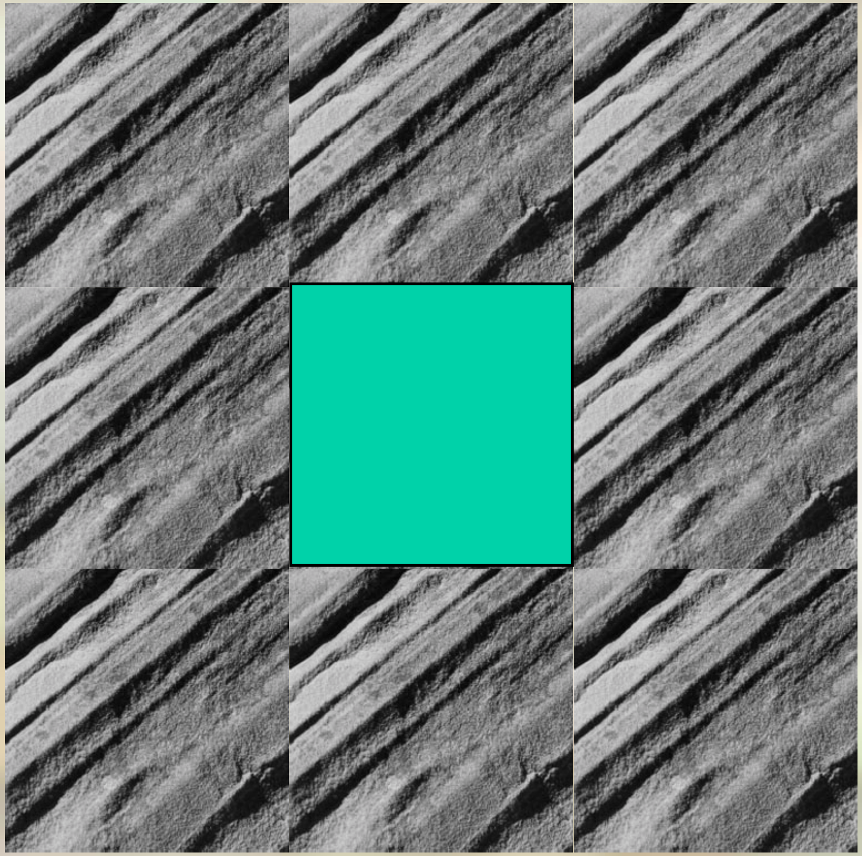
- The IDFT equation

$$\boldsymbol{f}[m,n] = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} \widetilde{F}[u,v] W_M^{-um} W_N^{-vn}$$

- Implies the periodic extension of the image as well:

$$\mathbf{f}(m + a\,M, n + b\,N) = \mathbf{f}(m,n)$$

Periodic extension of image



Centering the DFT

- Usually, the DFT is displayed with DC coordinate (u, v) = (0, 0) at the center.
- Then low frequency info (which dominates most images) will cluster at the center of the display.
- Centering is accomplished by taking the DFT of the alternating image:
- $[(-1)^{i+j} I(i,j) \; ; \; 0 \leq i, j \leq N-1]$
- This is for display only!

Centering the DFT

- Note that

$(-1)^{i+j} = e^{\pi\,(i+j)} = e^{\;N(i+j)/2} =$

so

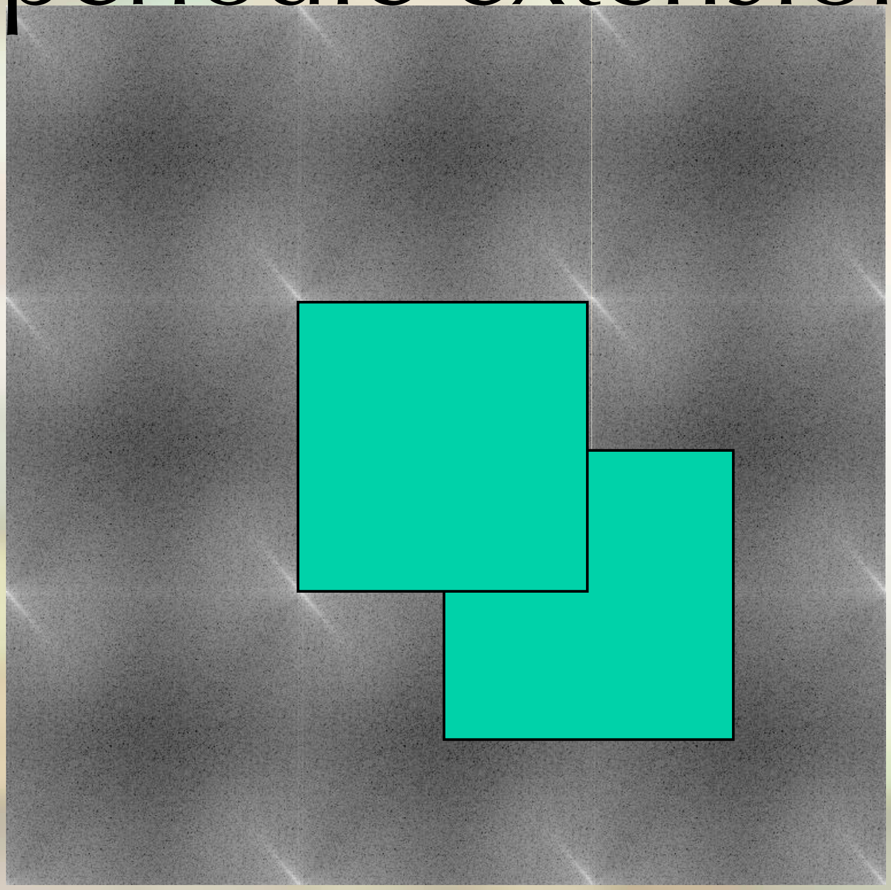
$$DFT[(-1)^{m+n}F[m,n] = \sum_{m,n=0}^{N-1} F[m,n](-1)^{m+n}W_N^{um+vn}$$

$$= \sum_{m,n=0}^{N-1} F[m,n]W_N^{-N(m+n)/2}W_N^{um+vn}$$

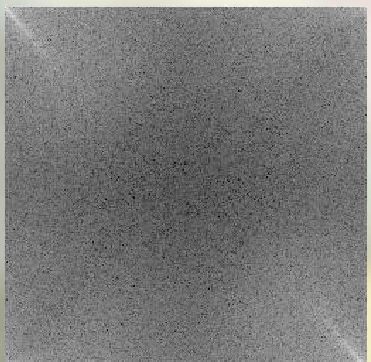
$$= \sum_{m,n=0}^{N-1} F[m,n]W_N^{(u-N/2)m+(v-N/2)n}$$

$$= \widetilde{F}[u-N/2, v-N/2]$$

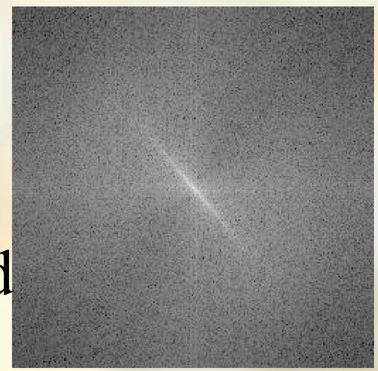
Shifted (centered) DFT from periodic extension

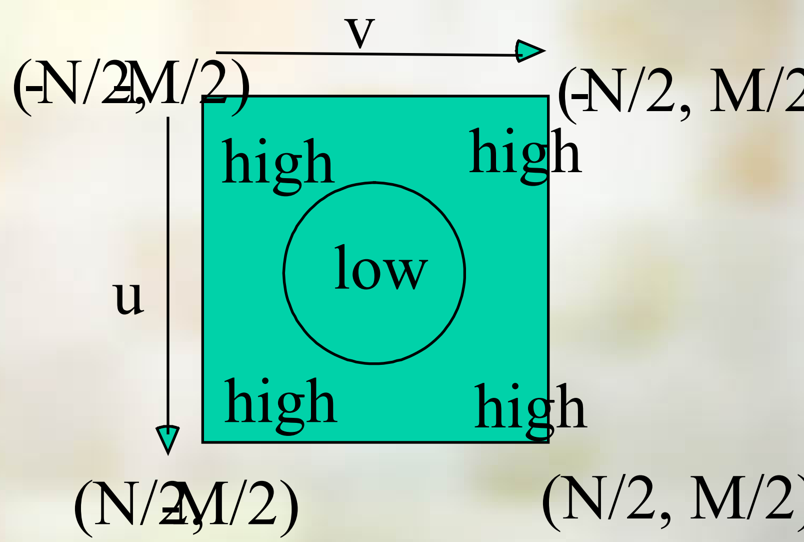


Centered DFT



centered





- DFT Example DEMO

Computation of the DFT

- Fast DFT algorithms collectively referred to as the Fast Fourier Transform (FFT).
- We won't study these – take a DSP class.
- Available in any math software library, Jitter!
- Forward and inverse DFTs essentially identical.
  Q. How are they different?

THE MEANING OF IMAGE FREQUENCIES

- It's easy to lose track of the meaning of the DFT and the notion of frequency content in all the math.

- By examining the DFT or spectrum of an image (especially its magnitude), we can often deduce much about the image.

QUALITATIVE PROPERTIES OF THE DFT

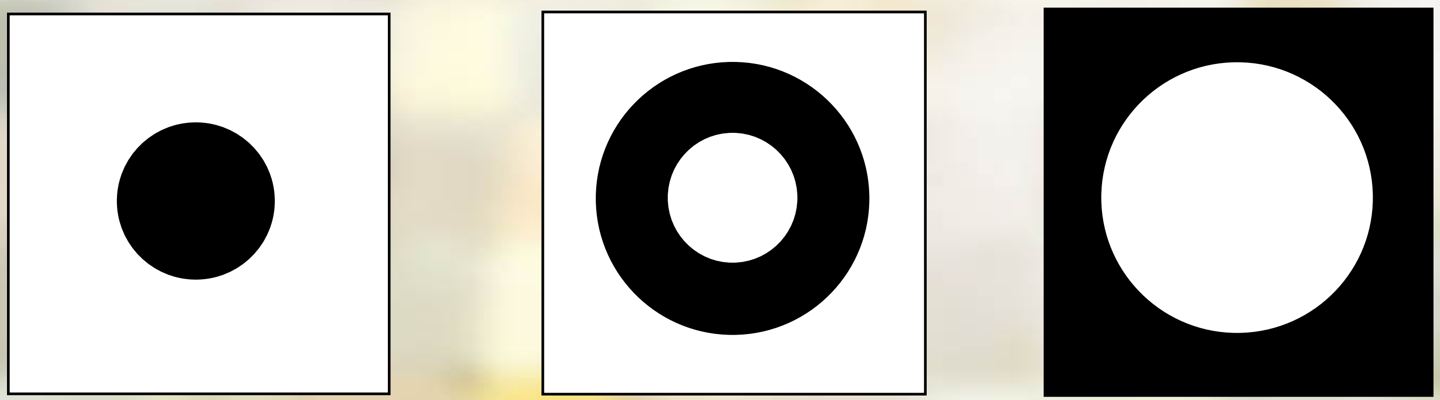
- We may regard the DFT magnitude as an image of frequency content.
- Bright regions in the DFT magnitude "image" correspond to frequencies having large magnitudes in the actual image.
- It is intuitive to think of image frequency content in terms of granularity (distribution of radial frequencies) and orientation.

IMAGE GRANULARITY

- Large DFT coefficients near the origin correspond to smooth image regions or a strong background. Since images are positive, image DFTs usually have a large peak at $(u, v) = (0, 0)$.
- The distribution of DFT coefficients relative to the origin is related to the granularity of the image.

MASKING DFT GRANULARITY

- Define toroidal zero-one masks (black = 1)



low-frequency mask   mid-frequency mask   high-frequency mask

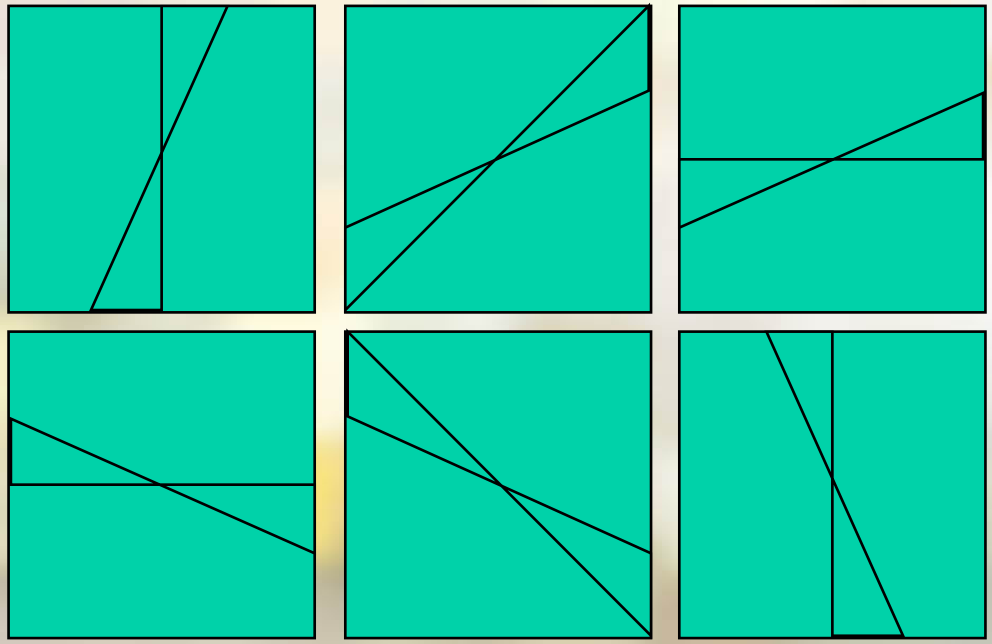
- Masking (multiplying) a DFT with these will produce IDFT images with only low-, middle-, or high frequencies: EXERCISE. Do this at home!

Image Directionality

- Large DFT coefficients along certain orientations correspond to highly directional image patterns.
- The distribution of DFT coefficients as a function of angle relative to the axes is related to the directionality of the image.

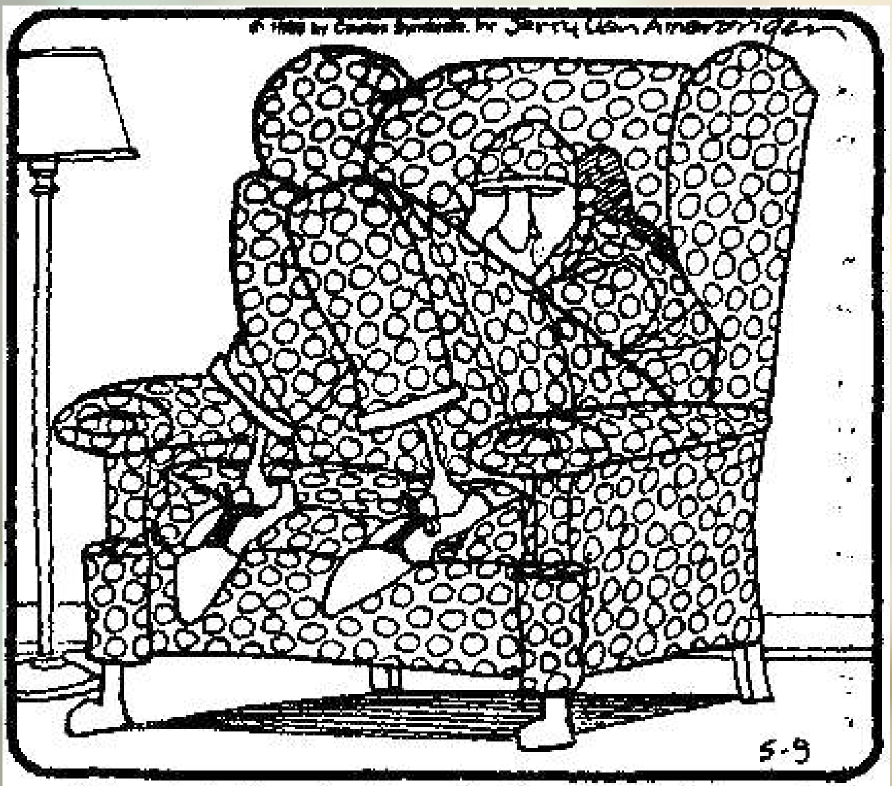
MASKING DFT GRANULARITY

- Try oriented, angular zero-one masks like these:



- The frequency origin is at the center of each mask.

Test Images



You get the feeling Bob's not going out and grabbing life by the throat anytime soon.

When the monster came, Lola, like the peppered moth and the arctic hare, remained motionless and undetected. Harold, of course, was immediately devoured.

- For Journal of Irreproducible Results

Aliased Chirp Image

- A chirp image

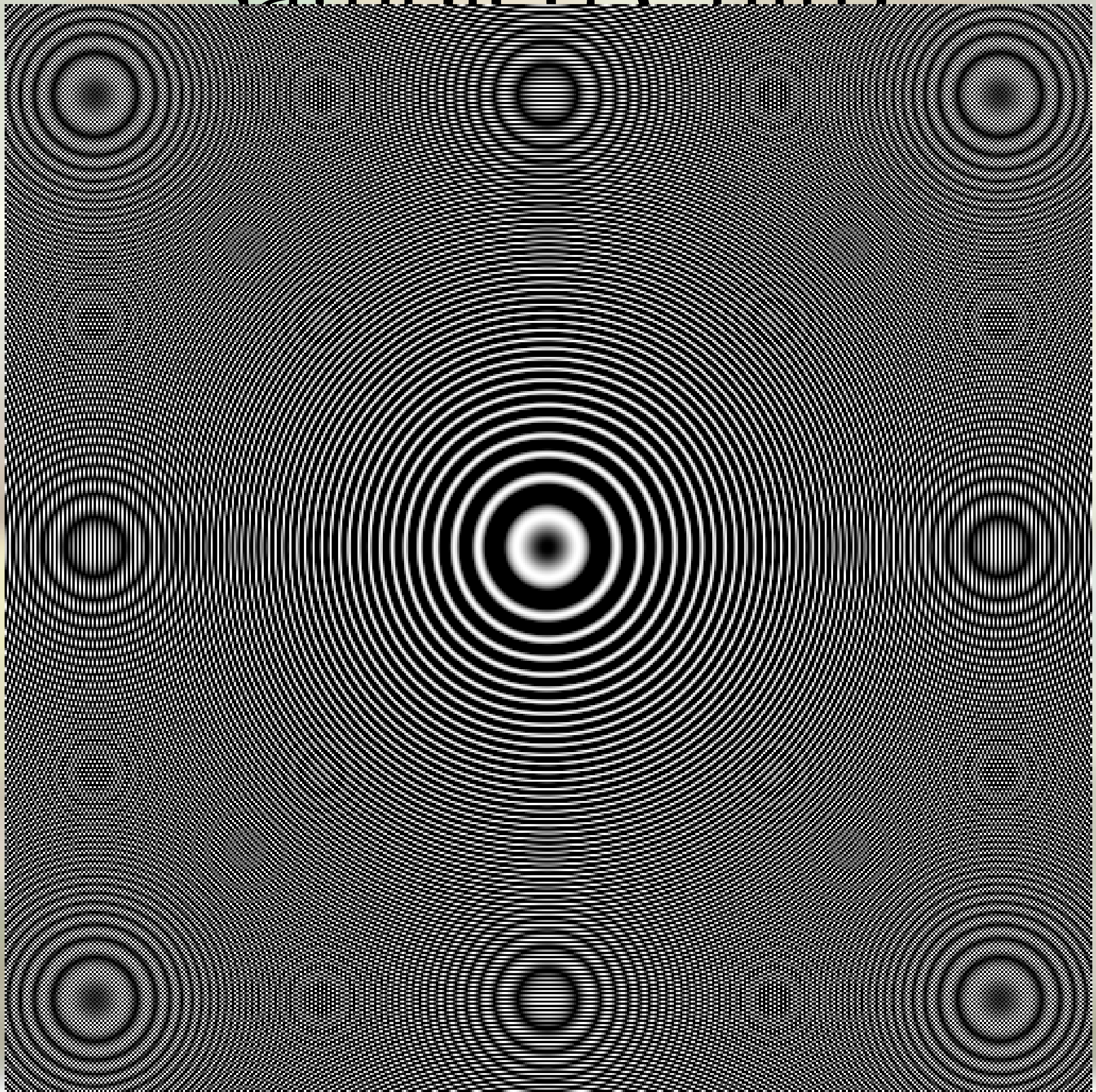
$$(x, y) = A\,(1+\cos(\,(x,y))) = A\,(1+\cos(ax \;+ by\;))$$

- has instantaneous spatial frequencies

$$(x, y) \quad (x,y) = 2\,(ax,\; by)$$

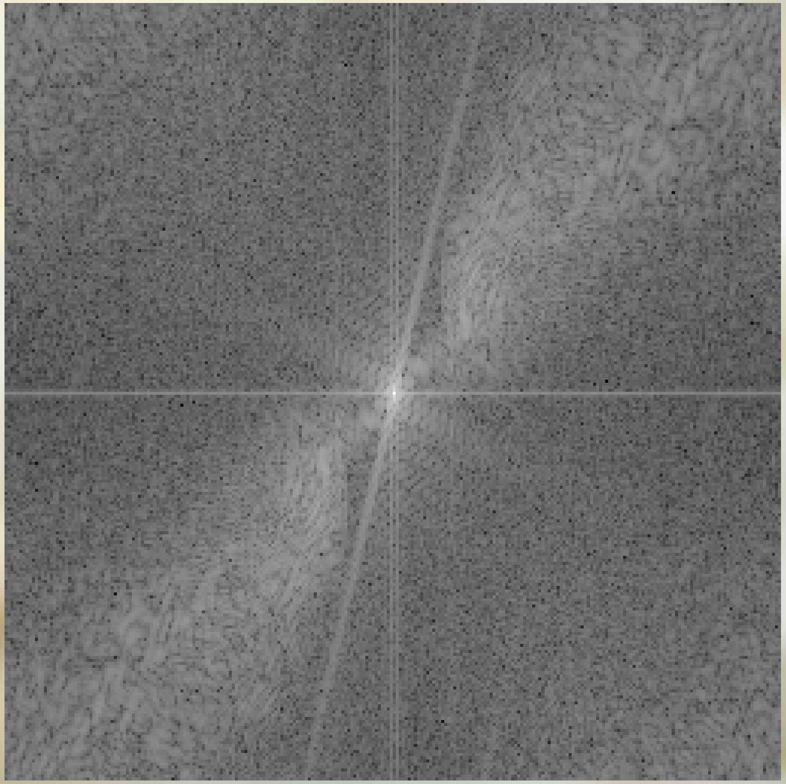
- which increase linearly away from the origin.

Sampled Chirp



Aliased Image

Sand Dune Image



Centered DFT Showing Aliasing

IMPORTANT 2-D FUNCTIONS AND THEIR DFTS

- It is worthwhile to examine the DFTs of some specific images. This is usually hard to do explicitly for the DFT / DSFT (infinite discrete image).
- So we'll give some simple ones.
- Then state some others as CFT transform pairs.

Constant Image

- If $\mathbf{f}(i, j) = c, \quad \text{for } 0 \le i \le N\text{-}1,\ 0 \le j \le M\text{-}1$
  Then

$$DFT[\mathbf{f}][u,v] = N^2 \cdot c \cdot \boldsymbol{\delta}(u, v)$$

where

- $\boldsymbol{\delta}(u, v)$ = unit impulse function
- = {1 at (u=0, v=0)
- 0 elsewhere }

2-D Unit Pulse Image

- Let $$\boldsymbol{F}[m,n] = c \cdot \delta[m,n]$$
- Then

$$\widetilde{F}[u,v] = \sum_{m,n=0}^{N-1} c \cdot \delta[m,n] W_N^{um+vn}$$

$$= c \cdot W_N^0 = c$$

Cosine Wave Image

- Let

$$F[m,n] = d \cdot cos[\pi(bm+cn)] = \frac{d}{2}[W_N^{bm+cn} + W_N^{-(bm+cn)}]$$

- by the Euler formula. Then

$$\widetilde{F}[u,v] = \frac{d}{2}\sum_{m,n=0}^{N-1}[W_N^{bm+cn} + W_N^{-(bm+cn)}]W_N^{um+vn}$$

$$= \frac{d}{2}\sum_{m,n=0}^{N-1}[W_N^{(u+b)m+(v+c)n} + W_N^{(u-b)m+(v-c)n}]$$

using the lemma DFT[impulse] = constant W_N^0

$$= \frac{d}{2}N^2[\delta(u+b,v+c) + \delta(u-b,v-c)]$$

so DFT is non-zero *only* at the frequencies of cosine wave

Sinusoidal Images

- Ditto for sine wave
- Sinusoids are concentrated single frequencies

Gaussian Function

- If

$$F_\sigma[m,n] = e^{-(m^2+n^2)/\sigma^2}$$

then

$$\widetilde{F_\sigma}[u,v] = e^{-2\pi^2\sigma^2(u^2+v^2)}$$

- The Fourier transform of a Gaussian is also Gaussian.

Comments

- We now have a basic understand of frequency-domain concepts
- We can put them to use in linear filtering applications.