This is an intellectually stimulating, informal presentation of those parts of point set topology that are of importance to the nonspecialist. In his presentation, and through many illustrations, the author strongly appeals to the intuition of the reader, presenting many examples and situations where the understanding of elementary topological questions will lead to much deeper and more advanced problems in topology and geometry.

ISBN 0-387-**90892**-7
ISBN 3-540-**90892**-7

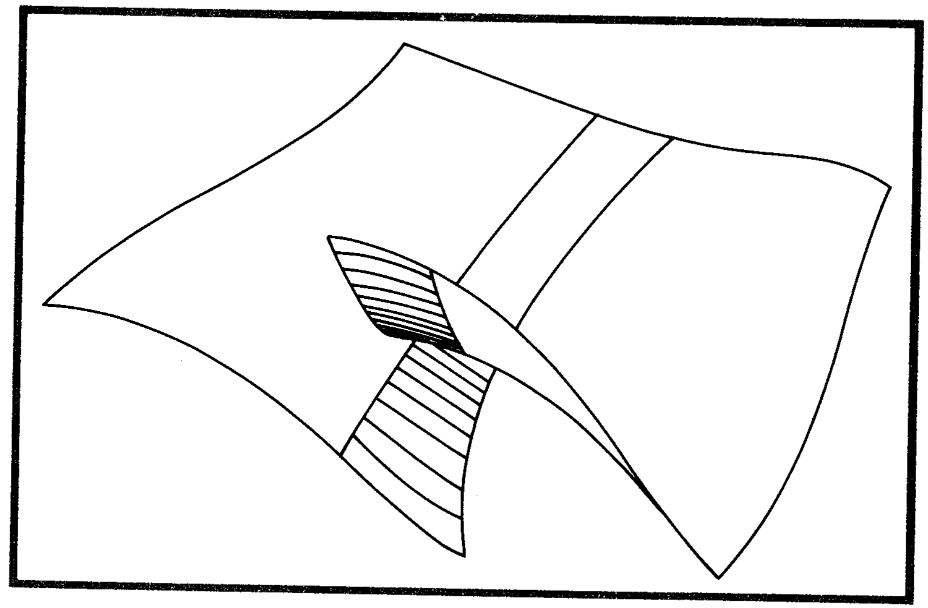
UTM

Jänich Topology

Undergraduate Texts in Mathematics

Klaus Jänich

Topology



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Klaus Jänich

Topology

Translated by Silvio Levy

With 181 Illustrations

Springer-Verlag
New York Berlin Heidelberg Tokyo

Klaus Jänich
Fakultät für Mathematik
Universität Regensburg
8400 Regensburg
Federal Republic of Germany

Silvio Levy (*Translator*)
Mathematics Department
Princeton University
Princeton, NJ 08544
U.S.A.

AMS Subject Classification: 54-01

Library of Congress Cataloging in Publication Data
Jänich, Klaus.
Topology.
(Undergraduate texts in mathematics)
Translation of: Topologie.
Bibliography: p.
Includes index.
1. Topology. I. Title. II. Series.
QA611.J3513 1984 514 83-14495

Original German edition: *Topologie*, Springer-Verlag, Berlin, Heidelberg, New York, © 1980.

Typeset by Composition House Ltd., Salisbury, England.
Printed and bound by R. R. Donnelley & Sons, Harrisonburg, Virginia.
Printed in the United States of America.

9 8 7 6 5 4 3 2 1

ISBN 0-387-90892-7 Springer-Verlag New York Berlin Heidelberg Tokyo
ISBN 3-540-90892-7 Springer-Verlag Berlin Heidelberg New York Tokyo

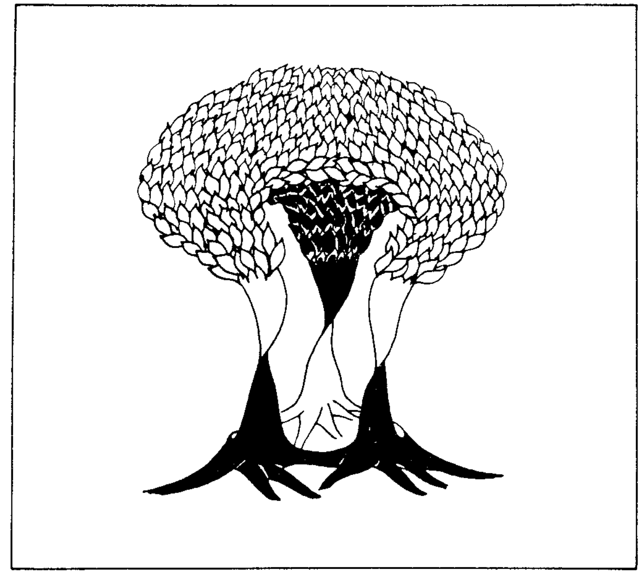
Preface

This volume covers approximately the amount of point-set topology that a student who does not intend to specialize in the field should nevertheless know. This is not a whole lot, and in condensed form would occupy perhaps only a small booklet. Our aim, however, was not economy of words, but a lively presentation of the ideas involved, an appeal to intuition in both the immediate and the higher meanings.

I wish to thank all those who have helped me with useful remarks about the German edition or the original manuscript, in particular, J. Bingener, Guy Hirsch and B. Sagraloff. I thank Theodor Bröcker for donating his "Last Chapter on Set-Theory" to my book; and finally my thanks are due to Silvio Levy, the translator. Usually, a foreign author is not very competent to judge the merits of a translation of his work, but he may at least be allowed to say: I like it.

Regensburg, May 1983 KLAUS JÄNICH

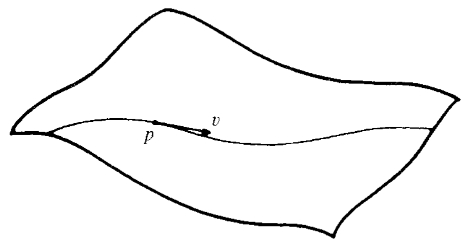
Introduction



§1. What Is Point-Set Topology About?

It is sometimes said that a characteristic of modern science is its high—and ever increasing—level of specialization; every one of us has heard the phrase "only a handful of specialists . . .". Now a general statement about so complex a phenomenon as "modern science" always has the chance of containing a certain amount of truth, but in the case of the above cliché about specialization the amount is fairly small. One might rather point to the great and ever increasing *interweaving* of formerly separated disciplines as a mark of modern science. What must be known today by, say, both a number theorist and a differential geometer, is much more, even relatively speaking, than it was fifty or a hundred years ago. This interweaving is a result of the fact that scientific development again and again brings to light hidden analogies whose further application represents such a great intellectual advance that the theory based on them very soon permeates all fields involved, connecting them together. Point-set topology is just such an analogy-based theory, comprising all that can be said in general about concepts related, though sometimes very loosely, to "closeness", "vicinity" and "convergence".

Theorems of one theory can be instruments in another. When, for instance, a differential geometer makes use of the fact that for each point and direction there is exactly one geodesic (which he does just about every day), he is



taking advantage of the Existence and Uniqueness Theorem for systems of second-order ordinary differential equations. On the other hand, the application of point-set topology to everyday uses in other fields is based not so much on deep theorems as on the unifying and simplifying power of its system of notions and of its felicitous terminology. And this power stems, in my understanding, from a very specific source, namely the fact that *point-set topology makes accessible to our spatial imagination a great number of problems which are entirely abstract and non-intuitive to begin with.* Many situations in point-set topology can be visualized in a perfectly adequate way in usual physical space, even when they do not actually take place there. Our spatial imagination, which is thus made available for mathematical reasoning about abstract things, is however a highly developed intellectual ability which is independent from abstraction and logical thinking; and this strengthening of our other mathematical talents is indeed the fundamental reason for the effectiveness and simplicity of topological methods.

§2. Origin and Beginnings

The emergence of fundamental mathematical concepts is almost always a long and intricate process. To be sure, one can point at a given moment and say: Here this concept, as understood today, is first defined in a clear-cut and plain way, from here on it "exists"—but by that time the concept had always passed through numerous preliminary stages, it was already known in important special cases, variants of it had been considered and discarded, etc., and it is often difficult, and sometimes impossible, to determine which mathematician supplied the decisive contribution and should be considered the originator of the concept in question.

In this sense one might say that the system of concepts of point-set topology "exists" since the appearance of Felix Hausdorff's book *Grundzüge der*

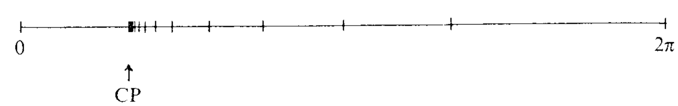
Mengenlehre (Leipzig, 1914). In its seventh chapter, "Point sets in general spaces", are defined the most important fundamental concepts of point-set topology. Maurice Fréchet, in his work "Sur quelques points du calcul fonctionnel" (*Rend. Circ. Mat. Palermo* **22**), had already come close to this mark, introducing the concept of metric spaces and attempting to grasp that of topological spaces as well (by axiomatizing the notion of convergence). Fréchet was primarily interested in function spaces and can perhaps be seen as the founder of the function analytic branch of point-set topology.

But the roots of the matter go, of course, deeper than that. Point-set topology, as so many other branches of mathematics, evolved out of the revolutionary changes undergone by the concept of geometry during the nineteenth century. In the beginning of the century the reigning view was the classical one, according to which geometry was the mathematical theory of the real physical space that surrounds us, and its axioms were seen as self-evident elementary facts. By the end of the century mathematicians had freed themselves from this narrow approach, and it had become clear that geometry was henceforth to have much wider aims, and should accordingly be made to work in abstract "spaces", such as n-dimensional manifolds, projective spaces, Riemann surfaces, function spaces etc. (Bolyai and Lobachevski, Riemann, Poincaré "and so on"—I'm not so bold as to try to delineate here this development process . . .). But now another contribution of paramount importance to the emergence of point-set topology was to be added to the rich variety of examples and the general ripeness to work with abstract spaces: namely, the work of Cantor. The dedication of Hausdorff's book reads: "To the creator of set theory, *Georg Cantor*, in grateful admiration."

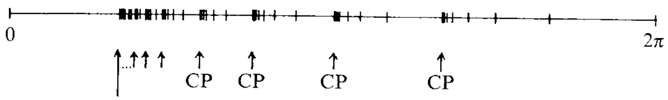
"A topological space is a pair consisting of a set and a set of subsets, such that . . ."—it is indeed clear that the concept could never have been grasped in such generality were it not for the introduction of abstract sets in mathematics, a development which we owe to Cantor. But long before establishing his transfinite set theory Cantor had contributed to the genesis of point-set in an entirely diverse way, about which I would like to add something.

Cantor had shown in 1870 that two Fourier series that converge pointwise to the same limit function have the same coefficients. In 1871 he improved this theorem by proving that the coefficients have to be the same also when convergence and equality of the limits hold for all points outside a finite exception set $A \subset [0, 2\pi]$. In a work of 1872 he now dealt with the problem of determining for which *infinite* exception sets uniqueness would still hold.

An infinite subset of $[0, 2\pi]$ must of course have at least one cluster point:



This is a very "innocent" example of an infinite subset of $[0, 2\pi]$. A somewhat "wilder" set would be one whose cluster points themselves cluster around some point:



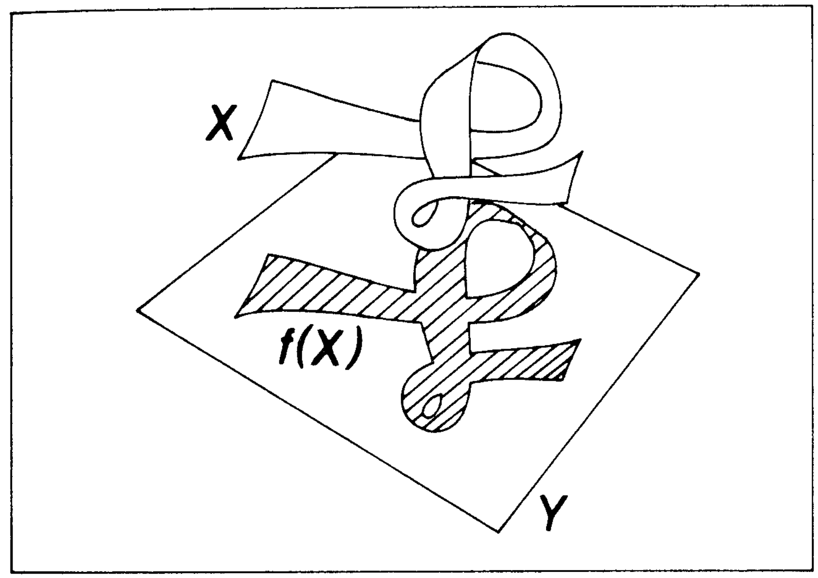
Cluster point of cluster points

Cantor now showed that if the sequence of subsets of $[0, 2\pi]$ defined inductively by $A^0 := A$ and $A^{n+1} := \{x \in [0, 2\pi] \mid x$ is a cluster point of $A^n\}$ breaks up after finitely many terms, that is if eventually we have $A^k = \varnothing$, then uniqueness *does* hold with A as the exception set. In particular a function that vanishes outside such a set (but not identically in the interval) cannot be represented by a Fourier series. This result helps to understand the strange convergence behavior of Fourier series, and the motivation for Cantor's investigation stems from classical analysis and ultimately from physics. But because of it Cantor was led to the discovery of a new type of subset $A \subset \mathbb{R}$, which must have been felt to be quite exotic, especially when the sequence $A, A^1, A^2, \ldots$ takes a *long* time to break off. Now the subsets of $\mathbb{R}$ move to the fore as objects to be studied in themselves, and, what is more, studied from what we would recognize today as being a topological viewpoint. Cantor continued along this path when later, while investigating general point sets in $\mathbb{R}$ and $\mathbb{R}^n$, he introduced the point-set topological approach, upon which Hausdorff could now base himself.

*

I do not want to give the impression that Cantor, Fréchet and Hausdorff were the only mathematicians to take part in the development and clarification of the fundamental ideas of point-set topology; but a more detailed treatment of the subject would be out of the scope of this book. I just wanted to outline, with a couple of sketchy but vivid lines, the starting point of the theory we are about to study.

CHAPTER I
Fundamental Concepts



§1. The Concept of a Topological Space

Definition. A *topological space* is a pair $(X, \mathcal{O})$ consisting of a set X and a set $\mathcal{O}$ of subsets of X (called "open sets"), such that the following axioms hold:

Axiom 1. Any union of open sets is open.

Axiom 2. The intersection of any two open sets is open.

Axiom 3. $\varnothing$ and X are open.

One also says that $\mathcal{O}$ is the *topology* of the topological space $(X, \mathcal{O})$. In general one drops the topology from the notation and speaks simply of a topological space X, as we'll do from now on:

Definition. Let X be a topological space.

(1) $A \subset X$ is called *closed* when $X \setminus A$ is open.
(2) $U \subset X$ is called a *neighborhood* of $x \in X$ if there is an open set V with $x \in V \subset U$.

(3) Let $B \subset X$ be any subset. A point $x \in X$ is called an *interior, exterior* or *boundary* (or *frontier*) *point of* B, respectively, according to whether B, $X \backslash B$ or neither is a neighborhood of x.
(4) The set $\mathring{B}$ of the interior points of B is called the *interior* of B.
(5) The set $\bar{B}$ of the points of X which are not exterior points of B is called the *closure* of B.

These are then the basic concepts of point-set topology; and the reader who is being introduced to them for the first time should at this point work out a couple of exercises, in order to become familiar with them. Once, when I was still a student at Tübingen, I was grading some exercises after a lecture on these fundamental concepts. In the lecture it had already been established that a set is open if and only if all of its points are interior, and one exercise went like this: Show that the set of interior points of a set is always open. In came a student asking why we had not accepted his reasoning: "The set of interior points contains only interior points (an indisputable tautology); hence, the problem is trivial." There were a couple of other graders present and we all zealously tried to convince him that in talking about interior points you have to specify what set they are interior to, but in vain. When he realized what we wanted, he left, calmly remarking that we were splitting hairs. What could we answer?

Therefore, should among my readers be a complete newcomer to the field, I would suggest him to verify right now that the interior of B is the union of all open sets contained in B, and that the closure of B is the intersection of all closed sets containing B. And as food for thought during a peaceful afternoon let me add the following remarks.

Each of the three concepts defined above using open sets, namely, "closed sets", "neighborhoods" and "closure", can in its turn be used to characterize openness. In fact, a set $B \subset X$ is open if and only if $X \backslash B$ is closed, if and only if B is a neighborhood of each of its points, and if and only if $X \backslash B$ is equal to its closure. Thus the system of axioms defining a topological space must be expressible in terms of each one of these concepts, for instance:

Alternative Definition for Topological Spaces (Axioms for Closed Sets). A topological space is a pair $(X, \mathscr{A})$ consisting of a set X and a set $\mathscr{A}$ of subsets of X (called "closed sets"), such that the following axioms hold:

A1. Any intersection of closed sets is closed.
A2. The union of any two closed sets is closed.
A3. X and ϕ are closed.

This new definition is equivalent to the old in that $(X, \mathcal{O})$ is a topological space in the sense of the old definition if and only if $(X, \mathscr{A})$ is one in the sense of the new, where $\mathscr{A} = \{X \backslash V \mid V \in \mathcal{O}\}$. Had we given the second definition first, closedness would have become the primary concept, openness following

by defining $X \backslash V$ to be open if and only if $V \subset X$ is closed. But the definition of concepts (2)–(5) would have been left untouched and given rise to the same system of concepts that we obtained in the beginning. It has become customary to start with open sets, but the idea of neighborhood is more intuitive, and indeed it was in terms of it that Hausdorff defined these notions originally:

Alternative Definition (Axioms for Neighborhood). A topological space is a pair $(X, \mathfrak{U})$ consisting of a set X and a family $\mathfrak{U} = \{\mathfrak{U}_x\}_{x \in X}$ of sets $\mathfrak{U}_x$ of subsets of X (called "neighborhoods of x") such that:

N1. Each neighborhood of x contains x, and X is a neighborhood of each of its points.
N2. If $V \subset X$ contains a neighborhood of x, then V itself is a neighborhood of x.
N3. The intersection of any two neighborhoods of x is a neighborhood of x.
N4. Each neighborhood of x contains a neighborhood of x that is also a neighborhood of each of its points.

One can see that these axioms are a bit more complicated to state than those for open sets. The characterization of topology by means of the closure operation, however, is again quite elegant and has its own name:

Alternative Definition (The Kuratowski Closure Axioms). A topological space is a pair $(X, ^{-})$ consisting of a set X and a map $^{-}: \mathfrak{P}(X) \to \mathfrak{P}(X)$ from the set of all subsets of X into itself such that:

C1. $\overline{\varnothing} = \varnothing$.
C2. $A \subset \bar{A}$ for all $A \subset X$.
C3. $\bar{\bar{A}} = \bar{A}$ for all $A \subset X$.
C4. $\overline{A \cup B} = \bar{A} \cup \bar{B}$ for all $A, B \in X$.

Formulating what exactly the equivalence of all these definitions means and then proving it is, as we said, left as an exercise to the new reader. We will stick to our first definition.

§2. Metric Spaces

As we know, a subset of $\mathbb{R}^n$ is called open in the usual topology when every point in it is the center of some ball also contained in the set. This definition can be extended in a natural way if instead of $\mathbb{R}^n$ we consider a set X for which the notion of distance is defined; in particular every such space gives rise to a topological space. Let's recall the following

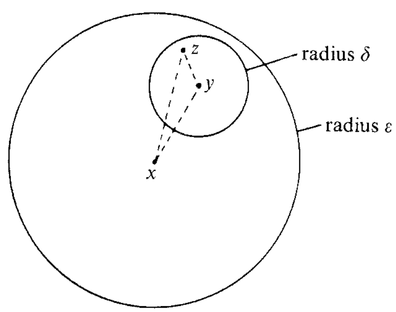
Definition (Metric Space). A metric space is a pair (X, d) consisting of a set X and a real function $d: X \times X \to \mathbb{R}$ (called the "metric"), such that:

M1. $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$.
M2. $d(x, y) = d(y, x)$ for all $x, y \in X$.
M3. (Triangle Inequality). $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Definition (Topology of a Metric Space). Let (X, d) be a metric space. A subset $V \subset X$ is called open if for every $x \in V$ there is an $\varepsilon > 0$ such that the "ε-ball" $K_\varepsilon(x) := \{y \in X \mid d(x, y) < \varepsilon\}$ centered at x is still contained in V. The set $\mathcal{O}(d)$ of all open sets of X is called the topology of the metric space (X, d).

Then $(X, \mathcal{O}(d))$ is really a topological space: and here again our hypothetic novice has an opportunity to practice. But at this point even the more experienced reader could well lean back on his chair, stare at the void and think for a few seconds about what role is played here by the triangle inequality.

So? Well, absolutely none. But as soon as we want to start doing something with these topological spaces $(X, \mathcal{O}(d))$, the inequality will become very useful. It allows us, for example, to draw the conclusion, familiar from $\mathbb{R}^n$, that around each point y such that $d(x, y) < \varepsilon$ there is a small δ-ball entirely contained in the ε-ball around x:



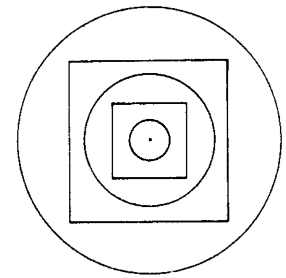
and consequently that the "open ball" $\{y \mid d(x, y) < \varepsilon\}$ is really open, whence in particular a subset $U \subset X$ is a neighborhood of x if and only if it contains a ball centered at x.

Metrics which are very different can in certain circumstances induce the same topology. If d and d' are metrics on X, and if every ball around x in the d metric contains a ball around x in the d' metric, we immediately have that every d-open set is d'-open, that is $\mathcal{O}(d) \subset \mathcal{O}(d')$. If furthermore the converse

also holds, then the two topologies are the same: $\mathcal{O}(d) = \mathcal{O}(d')$. An example is the case $X = \mathbb{R}^2$ and

$$d(x, y) := \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

$$d'(x, y) := \max\{|x_1 - y_1|, |x_2 - y_2|\}:$$



And here there is a simple but instructive trick that should be noted right from the start, a veritable talisman against false assumptions about the relationship between metric and topology: If (X, d) is a metric space, then so is (X, d'), where d' is given by $d'(x, y) := d(x, y)/(1 + d(x, y))$; moreover, as can be readily verified, $\mathcal{O}(d) = \mathcal{O}(d')$! But since all distances in d' are less than 1, it follows in particular that if a metric happens to be bounded this property can by no means be traced back to its topology.

Definition (Metrizable Spaces). A topological space $(X, \mathcal{O})$ is called *metrizable* if there is a metric d on X such that $\mathcal{O}(d) = \mathcal{O}$.

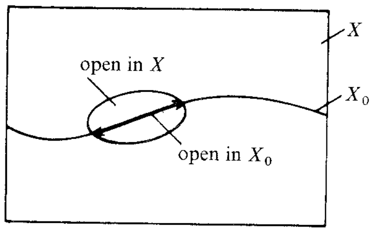
How can one determine whether or not a given topological space is metrizable? This question is answered by the "metrization theorems" of point-set topology. Are all but a few topological spaces metrizable, or is metrizability, on the contrary, a rare special case? The answer is neither, but rather the first than the second: there are a great many metrizable spaces. We will not deal with the metrization theorems in this book, but with the material in Chapters I, VI and VIII the reader will be quite well equipped for the further pursuit of this question.

§3. Subspaces, Disjoint Unions and Products

It often happens that new topological spaces are constructed out of old ones, and the three simplest and most important such constructions will be discussed now.

Definition (Subspace). If $(X, \mathcal{O})$ is a topological space and $X_0 \subset X$ a subset, the topology $\mathcal{O}|X_0 := \{U \cap X_0 | U \in \mathcal{O}\}$ on X_0 is called the *induced* or *subspace topology*, and the topological space $(X_0, \mathcal{O}|X_0)$ is called a *subspace* of $(X, \mathcal{O})$.

Instead of "open with respect to the topology of X_0" one says in short "open in X_0", and a subset $B \subset X_0$ is then open in X_0 if and only if it is the intersection of X_0 with a set open in X:



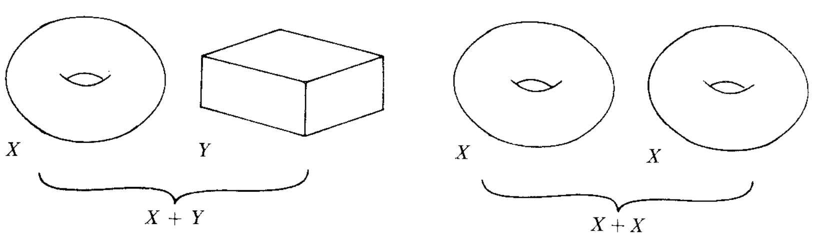
Thus such sets are not to be confused with sets "open and in X_0", since they *do not* have to be open—open, that is, in the topology of X.

Definition (Disjoint Union of Sets). If X and Y are sets, their *disjoint union* or *sum* is defined by means of some formal trick like for instance

$$X + Y := X \times \{0\} \cup Y \times \{1\}$$

—but we immediately start treating X and Y as subsets of $X + Y$, in the obvious way.

Intuitively this operation is nothing more than the disjoint juxtaposition of a copy of X and one of Y, and we obviously cannot write this as $X \cup Y$, since X and Y do not have to be disjoint to begin with, as for example when $X = Y$ and $X \cup X = X$ consists of only one copy of X.

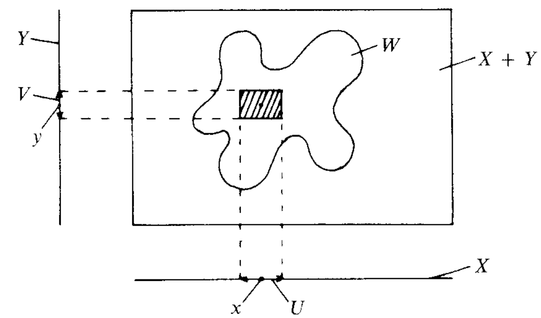


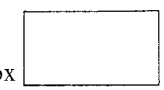
Definition (Disjoint Union of Topological Spaces). If $(X, \mathcal{O})$ and $(Y, \tilde{\mathcal{O}})$ are topological spaces, a new topology on $X + Y$ is given by

$$\{U + V | U \in \mathcal{O}, V \in \tilde{\mathcal{O}}\}$$

and the set $X + Y$ with this topology is called the *topological disjoint union* of the topological spaces X and Y.

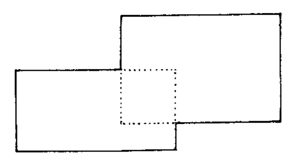
Definition (Product Topology). Let X and Y be topological spaces. A subset $W \subset X \times Y$ is called *open in the product topology* if for each point $(x, y) \in W$ there are neighborhoods U of x in X and V of y in Y such that $U \times V \subset W$. The set $X \times Y$ endowed with the above topology is called the (Cartesian) product of the spaces X and Y.



The box  is the usual mental image for the Cartesian product of sets or topological spaces, and as long as we are dealing with nothing too complicated, this image is perfectly adequate. I will call the products

$$U \times V \subset X \times Y$$

of open sets $U \subset X$ and $V \subset Y$ *open boxes*. Open boxes are obviously open in the product topology, but they are not the only open sets: by themselves they do not form a topology, since the union of two boxes is not in general a box:



This trivial observation would not have occurred to me if I had not often come upon the opposite, erroneous, opinion, which must possess some peculiar attraction.—Well, that's it for the time being.

§4. Bases and Subbases

Definition (Basis). Let X be a topological space. A set $\mathfrak{B}$ of open sets is called a basis for the topology if every open set is a union of sets in $\mathfrak{B}$.

For example, the open boxes form a basis for the product topology, and the open balls in $\mathbb{R}^n$ form a basis for the usual topology in $\mathbb{R}^n$; but notice that the set of balls with rational radius and rational center coordinates (which is countable!) is also a basis for the topology of $\mathbb{R}^n$.

Definition (Subbasis). Let X be a topological space. A set $\mathfrak{S}$ of open sets is called a subbasis for the topology if every open set is a union of finite intersections of sets in $\mathfrak{S}$.

Of course the word "finite" here does not mean that the intersection should be a finite set, but that it is the intersection of finitely many sets. This includes the intersection of zero sets (that is, an empty family of sets), which by a meaningful convention is defined to be equal to the whole space (since in this way the formula $\bigcap_{\lambda \in \Lambda} S_\lambda \cap \bigcap_{\mu \in M} S_\mu = \bigcap_{\nu \in \Lambda \cup M} S_\nu$ still holds). Analogously, the union of an empty family of sets is suitably defined as the empty set.

With these conventions we then have that if X is a set and $\mathfrak{S}$ an arbitrary set of parts of X, there is exactly one topology $\mathcal{O}(\mathfrak{S})$ on X such that $\mathfrak{S}$ is a subbasis for $\mathcal{O}(\mathfrak{S})$ (the topology "generated" by $\mathfrak{S}$). It consists exactly of the unions of finite intersections of sets in $\mathfrak{S}$.

Thus a topology can be defined by prescribing a subbasis. But why should one want to do it? Well, it often happens that one wants a topology satisfying certain conditions. Usually one of these conditions refers to the *fineness* of the topology. If $\mathcal{O}$ and $\mathcal{O}'$ are topologies on X, and if $\mathcal{O} \subset \mathcal{O}'$ one says that $\mathcal{O}'$ is *finer* than $\mathcal{O}$ and that $\mathcal{O}$ is *coarser* than $\mathcal{O}'$; and often there are reasons to look for a topology which is as fine or as coarse as possible. To be sure, there is a coarsest topology on X, the so-called *trivial* topology, which contains only the sets X and $\varnothing$; and there is a finest topology, the so-called *discrete* topology, in which all subsets of X are open. But this is not enough, for one wishes to impose other conditions as well. In a typical case, the desired topology should on the one hand be as coarse as possible, and on the other contain at least the sets of $\mathfrak{S}$. There is always such a topology: it is exactly our $\mathcal{O}(\mathfrak{S})$.

§5. Continuous Maps

Definition (Continuous Map). Let X and Y be topological spaces. A map $f: X \to Y$ is called continuous if the inverse image of open sets is always open.

Note. *The identity map* $\mathrm{id}_X : X \to X$ *is continuous, and if* $X \to Y$ *and* $Y \to Z$ *are continuous, so is* $g \circ f : X \to Z$.

With this the most important has been said. If the concept is new to you, I suggest two useful exercises for practice. The first consists in searching for the characterization of continuous maps in terms of the "alternative definitions" given in §1, that is, in verifying that a map $f: X \to Y$ is continuous if and only if the inverse image of any closed set is closed, if and only if the inverse image of a neighborhood is a neighborhood (more exactly, if U is a neighborhood of $f(x)$, then $f^{-1}(U)$ is a neighborhood of x), and if and only if $\overline{f^{-1}(B)} \subset f^{-1}(\bar{B})$ for all subsets $B \subset X$. Furthermore, considering the characterization of continuity in terms of neighborhoods in the special case of metric spaces leads to the good old "For every $\varepsilon > 0$ there is a $\delta > 0 \ldots$".

The second recommended exercise has to do with subspaces, disjoint unions and products, and consists in proving the following three notes:

Note 1. *If* $f: X \to Y$ *is continuous and* $X_0 \subset Y$ *is a subspace, then the restriction* $f | X_0 : X_0 \to Y$ *is also continuous.*
Note 2. $f : X + Y \to Z$ *is continuous if and only if* $f | X$ *and* $f | Y$ *are both continuous.*
Note 3. $(f_1, f_2): Z \to X \times Y$ *is continuous if and only if* $f_1 : Z \to X$ *and* $f_2 : Z \to Y$ *are both continuous.*

By the way, the properties stated in Notes 2 and 3 characterize the direct union and product topology.

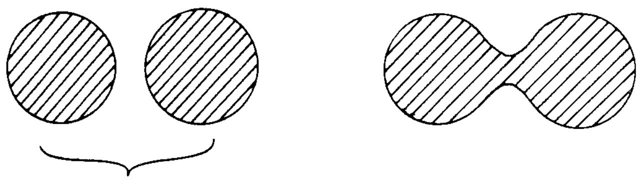
Definition (Homeomorphism). A bijective map $f: X \to Y$ is called a *homeomorphism* when both f and f^{-1} are continuous, that is when $U \subset X$ is open if and only if $f(U) \subset Y$ is.

Suppose a topological property (i.e. one that can be formulated in terms of open sets) holds for X or some subset $A \subset X$. Then, if f is a homeomorphism, the same property must hold for Y or the corresponding subset $f(A)$. For instance: $A \subset X$ is closed $\Leftrightarrow f(A) \subset Y$ is closed; $U \subset X$ is a neighborhood of $x \Leftrightarrow f(U)$ is a neighborhood of $f(x)$; $\mathfrak{B}$ is a basis for the topology on $X \Leftrightarrow \{f(B) | B \in \mathfrak{B}\}$ is a basis for the topology of Y; and so on. Thus homeomorphisms play the same role in topology that linear isomorphisms play in linear algebra, or that biholomorphic maps play in function theory, or group isomorphisms in group theory, or isometries in Riemannian geometry. For this reason we also use the notation $f: X \xrightarrow{\cong} Y$ for homeomorphisms, as well as $X \cong Y$ for homeomorphic spaces (i.e. spaces such that there is a homeomorphism from one to the other.)

Until now we have named very few topological properties of topological spaces. From the great many that there are, I have picked for this chapter on "fundamental concepts" three that are particularly important and widely different in character: connectedness, Hausdorffness and compactness. They will be discussed in the next three paragraphs.

§6. Connectedness

Definition (Connectedness). A topological space is called *connected* if it is not the union of two non-empty, open, disjoint subspaces; or, in other words, the whole space and the empty set are the only subsets which are at the same time open and closed.



Example. An (open, half-open, closed) interval $I \subset \mathbb{R}$ is always connected. Although simple, this example presents a special interest, since in many cases the connectedness of complicated spaces ultimately derives from that of the interval. We will thus repeat the proof succinctly: Suppose that $I = A \cup B$ and $A \cap B = \varnothing$, A and B both non-empty and open in the subspace topology of $I \subset \mathbb{R}$. Choose points $a \in A$, $b \in B$ (we can assume $a < b$). Let s be equal to $\inf\{x \in B \mid a < x\}$. Then every neighborhood of s contains points in B (by definition of infimum), but also points in A, for if s is not equal to a, then $a < s$ and $(a, s) \subset A$. Thus s cannot be a point of either A or B, which is a contradiction, since $s \in A \cup B$ and A and B are both open. qed.

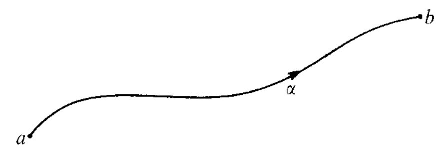
Example. The subspace $X = [0, 1] \cup (2, 3) \subset \mathbb{R}$ is not connected, because we can split it into the two non-empty open sets $A = [0, 1]$ and $B = (2, 3)$. (Objection: It is clear that $X = A \cup B$ and A and B are disjoint: but open? After all, A is a closed interval!! It may indeed be painful to have to call a closed interval open; but remember, folks, we're dealing with the topology of X and not that of $\mathbb{R}$!...)

What is this notion good for? Well, for one thing, it affords a crude way of distinguishing between topological spaces: if a space is connected and a second one is not, the two cannot be homeomorphic. Moreover, the following is also true: If X is a connected space, Y is a set and $f: X \to Y$ is locally constant (i.e. for each $x \in X$ there is a neighborhood U_x such that $f \mid U_x$ is constant), then f is constant over the whole domain. In fact, if y is a point in the image of f, $A = \{x \mid f(x) = y\}$ and $B = \{x \mid f(x) \neq y\}$ are both open, hence $X = A$ because X is connected, qed. This conclusion is often applied to the case $Y = \{\text{yes, no}\}$ or $\{\text{true, false}\}$, as follows: Let X be connected and let P be a property that points of X may or may not have, and suppose we want to prove that all points of X have property P. Then it is enough to prove the following three assertions:

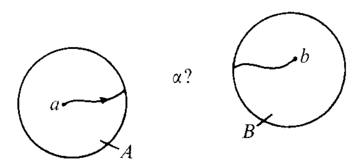
(1) There is at least *one* point with property P;
(2) If x has property P, the same applies to all points in a sufficiently small neighborhood;
(3) If x does not have property P, then the same applies to all points in a small neighborhood.

The following stronger concept is often of interest:

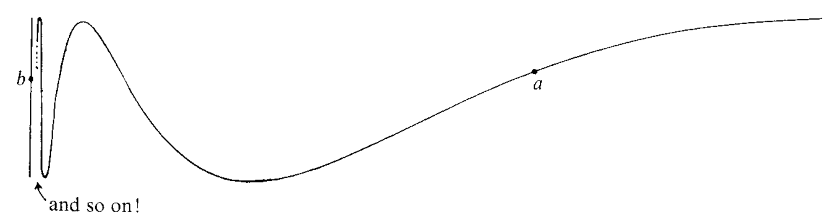
Definition (Path-Connectedness). X is said to be *path-connected* if every two points $a, b \in X$ are connected by a *path*, that is, a continuous map $\alpha: [0, 1] \to X$ such that $\alpha(0) = a$ and $\alpha(1) = b$:



One sees immediately that a path-connected space X is connected: If $X = A \cup B$, with A and B open, non-empty and disjoint, there can be no path from $a \in A$ to $b \in B$, due to the connectedness of $[0, 1]$ (otherwise we would have $[0, 1] = \alpha^{-1}(A) \cup \alpha^{-1}(B)$ and so on).

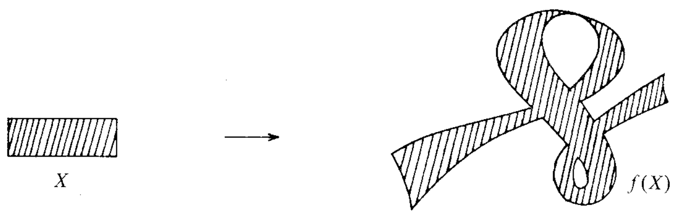


The converse is not true, though: a space can be connected and still manage to be "impassable" between two points. The subspace of $\mathbb{R}^2$ given by $\{(x, \sin \ln x) \mid x > 0\} \cup (0 \times [-1, 1])$ is an example:

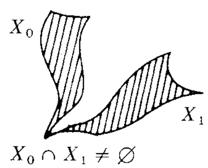


To conclude let me add three remarks concerning the behavior of connectedness under different operations. Topological properties such as connectedness tend to acquire, upon closer acquaintance, emotional overtones: some appear friendly and helpful, after we have seen several times how they make proofs easy or even possible in the first place; others, on the contrary, we come to dread, for the exactly opposite reason. True enough, a property of good repute can on occasion be an obstacle, and many properties are entirely ambivalent. But I can assure you that connectedness, Hausdorffness and compactness are predominantly "good" properties, and one would naturally like to know if such good properties are transferred from the building blocks to the final products by the usual topological constructions and processes. Thus:

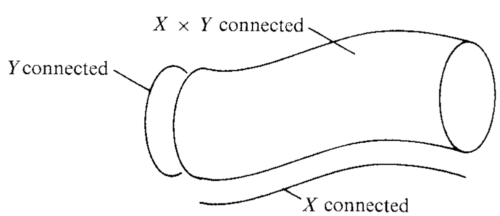
Note 1. *Continuous images of (path-)connected spaces are (path-)connected. In other words, if X is (path-)connected and $f: X \to Y$ is continuous, then the subspace $f(X)$ of Y is also (path-)connected. For a decomposition of $f(X)$ as $A \cup B$ would imply the same for $X = f^{-1}(A) \cup f^{-1}(B)$, etc.*



Note 2. *Non-disjoint unions of (path-)connected spaces are (path-)connected, that is if X_0 and X_1 are (path-)connected subspaces of X with $X = X_0 \cup X_1$ and $X_0 \cap X_1 \neq \varnothing$, then X is (path-)connected.*



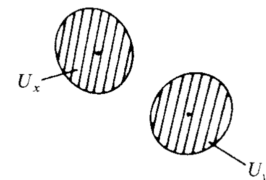
Note 3. *A Cartesian product $X \times Y$ of non-empty topological spaces X and Y is (path-)connected if and only if both factors are.*



Facetious question: How about the disjoint union X and Y?

§7. The Hausdorff Separation Axiom

Definition (Hausdorff Separation Axiom). A topological space is called Hausdorff if for any two different points there exist disjoint neighborhoods.



For example, every metric space is Hausdorff, for if d is a metric and $d(x, y) = \varepsilon > 0$, then the sets

$$U_x := \{z \mid d(x, z) < \varepsilon/2\} \quad \text{and} \quad U_y := \{z \mid d(y, z) < \varepsilon/2\},$$

for instance, are disjoint neighborhoods.

The property "non-Hausdorff" is quite counterintuitive and at first glance even unreasonable, seeming to go against our intuition of the neighborhood concept. For this reason Hausdorff included the above separation axiom in his original definition of "topological space" (1914). But later it was found that non-Hausdorff topologies too can be very useful, e.g. the "Zariski topology" in algebraic geometry. In any case one can step fairly deep into topology without really feeling a need for non-Hausdorff spaces, though here and there it is more convenient not to have to watch for Hausdorffness. For those who want to see such an exotic thing once, take a set X with more than one element and consider on it the trivial topology $\{X, \varnothing\}$.

One of the advantages offered by the separation axiom is the uniqueness of convergence:

Definition (Convergent Sequence). Let X be a topological space, $(x_n)_{n \in \mathbb{N}}$ a sequence in X. A point $a \in X$ is called limit of the sequence if for every neighborhood U of a there is an n_0 such that $x_n \in U$ for all $n \geq n_0$.

Note. *In a Hausdorff space a sequence can have at most one limit.*

In a trivial topological space, on the other hand, every sequence converges to every point.

As for behavior under operations, we note the following easily proved fact:

Note. *Every subspace of a Hausdorff space is Hausdorff, and two non-empty topological spaces X and Y are Hausdorff if and only if their disjoint union $X + Y$ is and if and only if their product $X \times Y$ is.*

*

The Hausdorff separation axiom is also called T_2. This sounds like there is a T_1, doesn't it? Well, how about this: $T_0, T_1, T_2, T_3, T_4, T_5$, not to mention $T_{2\frac{1}{2}}$ and $T_{3\frac{1}{2}}$! The Hausdorff axiom, however, is by far the most important of these, and deserves most to be kept in mind. Shall I say what T_1 stands for ...? But no. We can wait for it.

§8. Compactness

Ah, compactness! A wonderful property. This is true especially in differential and algebraic topology, as a rule, because everything works much more smoothly, easily and fully when we are dealing with compact spaces, manifolds, CW-complexes, groups etc. Now not everything in the world can be compact, but even for "non-compact" problems the compact case is often a good first step: We must first master the "compact terrain", which is easier to conquer, and then work our way into the non-compact case with modified techniques. Exceptions confirm the rule: Occasionally non-compactness also offers advantages, there is more "room" for certain constructions.... But now:

Definition (Compactness). A topological space is called compact if every open cover possesses a finite subcover. This means that X is compact if the following holds: If $\mathfrak{U} = \{U_\lambda\}_{\lambda\in\Lambda}$ is an arbitrary open cover of X, i.e. $U_\lambda \subset X$ open and $\bigcup_{\lambda\in\Lambda} U_\lambda = X$, then there are a finite number of $\lambda_1, \dots, \lambda_r \in \Lambda$ such that $U_{\lambda_1} \cup \cdots \cup U_{\lambda_r} = X$.

(**Remark.** Many authors call such spaces "quasicompact" and save the word "compact" for "quasicompact and Hausdorff".)

In compact spaces the following type of generalization from "local" to "global" properties is possible: Let X be a compact space and P a property that the open subsets of X may or may not have, and also such that if U and V have it, then so does $U \cup V$. (Examples below.) Then if X has this property *locally*, i.e. every point has a neighborhood with property P, then X itself has property P. In fact, such open neighborhoods form an open cover $\{U_x\}_{x\in X}$ of X; but, choosing the x_i appropriately, we have

$$X = U_{x_1} \cup \cdots \cup U_{x_r},$$

and by assumption the property is inductively transferred to finite unions, qed.

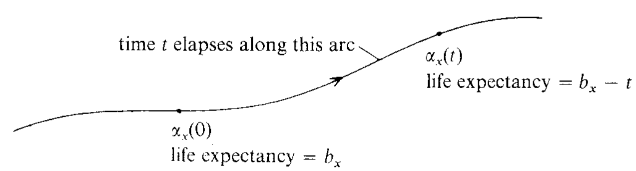
Example 1. Let X be compact and $f: X \to \mathbb{R}$ locally bounded (continuous, for example). Then f is bounded.

Example 2. Let X be compact and $(f_n)_{n\geq 1}$ a locally uniformly convergent sequence of functions on X. Then the sequence converges uniformly over the whole of X.

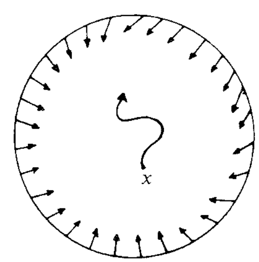
Example 3. Let X be compact and $\{A_\lambda\}_{\lambda\in\Lambda}$ a locally finite cover (i.e. each point has a neighborhood that intersects A_λ for only finitely many λ); then the cover is finite.

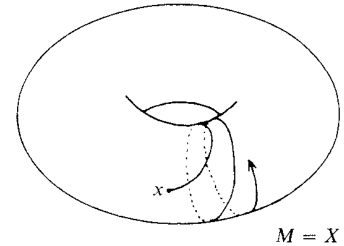
Example 4. Let X be compact and $A \subset X$ a locally finite subset (supply definition). Then A is finite. Or, conversely, if $A \subset X$ is infinite, there is a point $x \in X$ all of whose neighborhoods contain infinitely many points of A.

Example 5. Let v be a differentiable vector field on a manifold M, for instance an open set of $\mathbb{R}^n$. Denote by $\alpha_x: (a_x, b_x) \to M$ the maximal integral curve with $\alpha(0) = x$ and, reasonably enough, call b_x the (remaining) life expectancy and $-a_x > 0$ the age of x under v. From the local theory of ordinary differential equations it follows that locally there are positive lower bounds for life expectancy and age. Thus—and here comes in the compactness—there are such lower bounds for any compact set $X \subset M$ as well. Now as a point moves forward along its solution curve, its age increases and its life expectancy decreases:



If the life expectancy were finite, $b_x < \infty$, then it would eventually become as small as desired, and we obtain the well-known and useful lemma: *If a point in a compact subspace $X \subset M$ has finite life expectancy, it must use it before it is over to abandon X forever.* What then if there is no possibility for a point to abandon X—whether because the boundary of X is barricaded with vectors that point inwards all the time, or because the whole universe M is compact and $X = M$?





Then every point of X must move forever; and in particular a vector field on a compact manifold without boundary is always globally integrable.

But back to our subject! The consequences of this possibility of passing from local to global cannot, of course, be exhausted in a few pages, but I wanted to illustrate a bit, and not only to state, the usefulness of the notion of compactness.

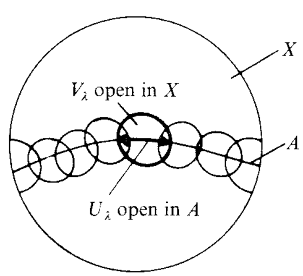
Examples of compact spaces? The closed interval $[0, 1]$ is an unpretentious but important example, because from it many others derive. It is well known that for every open cover of $[0, 1]$ there is a "Lebesgue number", i.e. a $\delta > 0$ such that every subinterval of length δ lies in one of the sets of the cover. (If there were not such a number, one could choose a sequence $(I_n)_{n\geq 1}$ of subintervals $I_n \subset [0, 1]$ with length $1/n$ none of which is contained in any of the sets of the cover. There must be a subsequence of the sequence of mid-points of the I_n converging to an $x \in [0, 1]$; but since x is in some set of the cover, we get a contradiction for n large.) Now since $[0, 1]$ can be covered by finitely many intervals of length δ, it can also be covered by finitely many sets of the open cover.

Proposition 1. *Continuous images of compact spaces are compact, or in other words, if X is a compact space and $f: X \to Y$ is continuous, then $f(X)$ is a compact subspace of Y.*

Proof. Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be an open cover of $f(X)$. Then $\{f^{-1}(U_\lambda)\}_{\lambda\in\Lambda}$ is an open cover of X, hence $X = f^{-1}(U_{\lambda_1}) \cup \cdots \cup f^{-1}(U_{\lambda_r})$ with an appropriate choice of indices, hence $f(X) = U_{\lambda_1} \cup \cdots \cup U_{\lambda_r}$, qed. □

Proposition 2. *Closed subspaces of compact spaces are compact.*

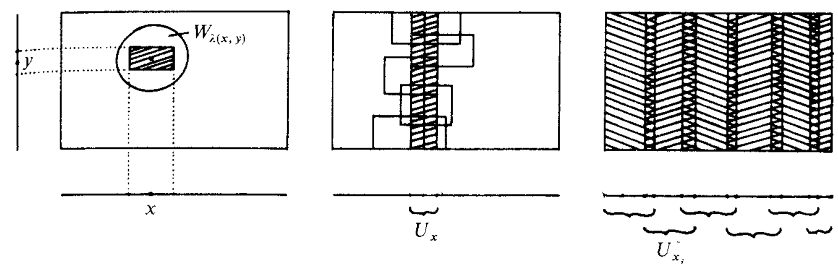
Proof. Let X be compact, $A \subset X$ closed, $\{U_\lambda\}_{\lambda\in\Lambda}$ an open cover of A. By the definition of subspace topology there is then a family $\{V_\lambda\}_{\lambda\in\Lambda}$ of sets open in X such that $U_\lambda = A \cap V_\lambda$:



Now since A is closed, $\{X\setminus A, \{V_\lambda\}_{\lambda\in\Lambda}\}$ is an *open* cover for X, hence there are $\lambda_1, \ldots, \lambda_r$ with $(X\setminus A) \cup V_{\lambda_1} \cup \cdots \cup V_{\lambda_r} = X$, i.e., $U_{\lambda_1} \cup \cdots \cup U_{\lambda_r} = A$, qed. □

Proposition 3. *Two non-empty spaces X and Y are both compact if and only if their disjoint union is, and if and only if their product is.*

Proof. (We'll prove only that the product of compact spaces is compact, which is the most interesting and relatively more difficult assertion. The converse follows from Proposition 1, and the statement about the disjoint union is trivial.) Let X and Y be compact and $\{W_\lambda\}_{\lambda\in\Lambda}$ an open cover of $X \times Y$.



Step 1 Step 2 Step 3

Step 1. We can choose for each (x, y) a $\lambda(x, y)$ such that $(x, y) \in W_{\lambda(x,y)}$, and because $W_{\lambda(x,y)}$ is open it contains an open box $U_{(x,y)} \times V_{(x,y)}$ around (x, y).

Step 2. For a fixed x the family $\{V_{(x,y)}\}_{y\in Y}$ is an open cover of Y, hence there are $y_1(x), \ldots, y_{r_x}(x)$ such that

$$V_{(x,y_1(x))} \cup \cdots \cup V_{(x,y_{r_x}(x))} = Y$$

Now put

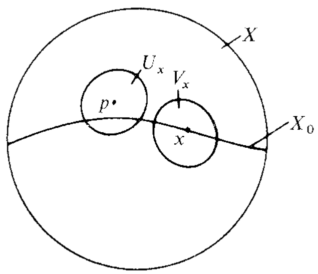
$$U_{(x,y_1(x))} \cap \cdots \cap U_{(x,y_{r_x}(x))} =: U_x.$$

Step 3. Since X is compact, there are $x_1, \ldots, x_n$ with $U_{x_1} \cup \cdots \cup U_{x_n} = X$, and consequently $X \times Y$ is covered by the (finitely many!) $W_{\lambda(x_i, y_j(x_i))}$, $1 \leq i \leq n$, $1 \leq j \leq r_i$, qed. □

From the compactness of the closed interval and these three propositions we can prove the compactness of many other spaces, e.g. all closed subspaces of the n-dimensional cube and hence all closed and bounded subsets of $\mathbb{R}^n$. This is one half of the famous Heine–Borel theorem, which states that a subset of $\mathbb{R}^n$ is compact *if and only if* it is closed and bounded. Why is every compact subset X_0 of $\mathbb{R}^n$ closed and bounded? Well, we have already observed that continuous functions on compact sets are bounded, and this applies in particular to the norm function, hence X_0 is bounded. As for closedness, it follows from the following simple but useful

Lemma. *If X is a Hausdorff space and $X_0 \subset X$ a compact subspace, then X_0 is closed in X.*

Proof. We must show that $X \setminus X_0$ is open, hence that every point p has a neighborhood U that does not intersect X_0. For each $x \in X_0$ choose disjoint neighborhoods U_x of p and V_x of x. It may happen that U_x intersects X_0,



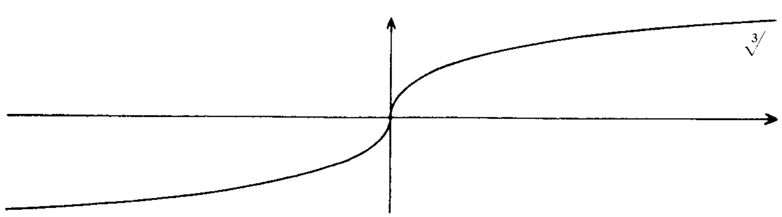
but at least it does not intersect the subset $V_x \cap X_0$, and if we now choose finitely many points $x_1, \ldots, x_n \in X_0$ such that

$$(V_{x_1} \cap X_0) \cup \cdots \cup (V_{x_n} \cap X_0) = X_0$$

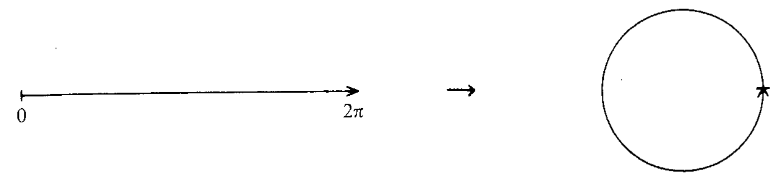
(which is always possible because of compactness), then $U := U_{x_1} \cap \cdots \cap U_{x_n}$ is a neighborhood of p with the desired property of not intersecting X_0, qed. □

*

Last but not least, I will present a nice little theorem about homeomorphisms, but first a few words to put it in the proper light. The first notions of isomorphism are introduced to us in linear algebra, and to prove that a linear map $f: V \to W$ is an isomorphism, it is enough to verify bijectivity, because $f^{-1}: W \to V$ is then automatically linear. The same applies for instance to groups and group homomorphisms. Having got accustomed to that, it is with a certain chagrin that we realize that there are other nice properties of bijections which are not transferred to the inverse: for instance, $x \mapsto x^3$ defines a differentiable bijection from $\mathbb{R}$ into $\mathbb{R}$, but the inverse map is not differentiable at the origin:



Unfortunately it is no better with continuity: take for instance the identity map from a set X with the discrete topology to X with the trivial topology. Nor does one have to resort to such extreme examples: Just wrap the half-open interval $[0, 2\pi)$ once around the unit circle, using the function $t \mapsto e^{it}$,



and we have a continuous bijection which cannot be a homeomorphism, because the circle is compact and the half-open interval is not. But even when f^{-1} *is* continuous, establishing this fact can turn out to be quite troublesome, especially when the continuity of f itself is obtained from an explicit formula $y = f(x)$, and there seems to be no way to write out a corresponding formula $x = f^{-1}(y)$. For this reason it is useful to have a condition, general in character and often satisfied, under which the inverse of a continuous bijection is always continuous:

Theorem. *A continuous bijection $f: X \to Y$ from a compact space X into a Hausdorff space Y is always a homeomorphism.*

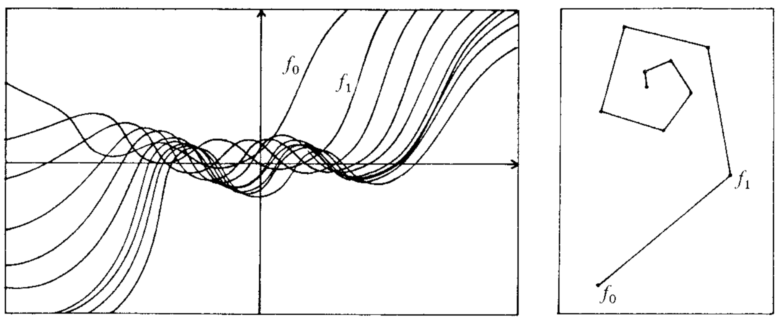
Proof. We have to show that the images of open sets are open, or, equivalently, that the images of closed sets are closed. Let then $A \subset X$ be closed. Then A is compact, since it is a closed subspace of a compact space; this means $f(A)$ is compact (continuous image of a compact space) and hence closed (compact subspace of the Hausdorff space Y), qed. □

CHAPTER II

Topological Vector Spaces

A large number of elements which intervene in mathematics are each completely determined by an infinite series of real or complex numbers: For example, a Taylor series is determined by the sequence of its coefficients . . . One can thus consider the numbers of the sequence which determine each of the elements as the coordinates of this element seen as a point of a space (E_ω) having a countably infinite number of dimensions. There are several advantages to working thus. First, the advantage that always appears when we use geometrical language, which favors intuition because of the analogies that it gives rise to . . .

MAURICE FRÉCHET
On Some Points of Functional Calculus (1906)



§1. The Notion of a Topological Vector Space

The present short chapter aims at nothing higher than presenting a certain class of examples of topological spaces, which really occur within the range of application of topology (in this case in functional analysis), and which is in fact of great significance: the topological vector spaces. It is only fair to place these examples right in the beginning, as they have played an important role in the formation of the notion of topological spaces (Fréchet 1906).

Definition (Topological Vector Space). Let $\mathbb{K} := \mathbb{R}$ or $\mathbb{C}$. A $\mathbb{K}$-vector space E with a topological space structure is called a *topological vector space* if its topological and linear structure are compatible in the following sense:

Axiom 1. The subtraction $E \times E \to E$ is continuous.
Axiom 2. Multiplication by scalars $\mathbb{K} \times E \to E$ is continuous.

Remark. Some authors impose an additional

Axiom 3. E is Hausdorff (e.g. Dunford–Schwartz [7]; but not Bourbaki [1]).

Instead of the subtraction we might as well have required the addition to be continuous, because it follows from Axiom 2 that the map $E \to E, x \mapsto -x$ is continuous, hence so is $E \times E \to E \times E$, $(x, y) \mapsto (x, -y)$. But there is one reason to phrase Axiom 1 with "subtraction" instead of "addition", and this reason, which I'll presently explain, is none the worse for being purely esthetic.

In the same way that there is a connection here between the notions of "vector space" and "topological space", so also many other interesting and useful concepts arise from a connection between the topology and the algebraic structure. In particular a group G which is also a topological space will be called a *topological group* if the group structure and the topology are "compatible". And what will be meant by that? Well, that the composition

$$G \times G \to G, \qquad (a, b) \mapsto ab$$

and the inverse map $G \to G$, $a \to a^{-1}$ are continuous. But these two conditions can be merged into one, the axiom for topological groups: The map $G \times G \to G$, $(a, b) \mapsto ab^{-1}$ is continuous.

Thus Axiom 1 says exactly that the additive group $(E, +)$ together with the topology of E forms a topological group.

In the next four paragraphs we will introduce the most common classes of topological vector space, in order of increasing generality.

§2. Finite-Dimensional Vector Spaces

$\mathbb{K}^n$, with the usual topology, is a topological vector space, and every isomorphism $\mathbb{K}^n \to \mathbb{K}^n$ is also a homeomorphism. Thus every n-dimensional vector space V has exactly one topology for which some (and consequently any) isomorphism $V \cong \mathbb{K}^n$ is a homeomorphism, and with this topology V becomes a topological vector space. This is all trivial, and undoubtedly the "usual" topology defined in this way is the most obvious one could find for V. But this topology is in fact more than just "obvious", for we have the following

Theorem (no proof given here, see, for instance, Bourbaki [1], Th. 2, p. 18). *The usual topology on a finite-dimensional vector space V is the only one that makes it into a Hausdorff topological vector space.*

The theorem shows that finite-dimensional topological vector spaces as such are not interesting, and the notion has been introduced because of the infinite-dimensional case. But even for these the theorem has an important consequence: namely, if V is a finite-dimensional vector subspace of any Hausdorff topological vector space E, then the topology on V induced from E is exactly the usual topology—even if E is one of the wilder specimens of its category.

§3. Hilbert Spaces

Let's recall that an *inner product space* is a real (resp. complex) vector space E together with a symmetric (resp. Hermitian) positive definite bilinear form $\langle .., .. \rangle$. Then for $v \in E$ the scalar $\|v\| := \sqrt{\langle v, v \rangle}$ is called the norm of v.

Note. *If $(E, \langle .., .. \rangle)$ is an inner product space, $d(v, w) := \|v - w\|$ defines a metric whose topology makes E into a topological vector space.*

Definition (Hilbert space). An inner product space is called a *Hilbert space* when it is complete relative to its metric, i.e. when every Cauchy sequence converges.

Hilbert spaces are surely, after finite-dimensional spaces, the most innocent topological vector spaces, and they can be completely classified, as follows: A family $\{e_\lambda\}_{\lambda \in \Lambda}$ of pairwise orthogonal unit vectors in a Hilbert space is called a *Hilbert basis* for H if the only vector orthogonal to all the e_λ is the zero vector. It can be proved that every Hilbert space has such a basis, any two bases of the same Hilbert space have the same cardinality, and finally two Hilbert spaces with equipotent bases are isometrically isomorphic.

§4. Banach Spaces

Definition (Normed Spaces). Let E be a $\mathbb{K}$-vector space. A map $\|..\|: E \to \mathbb{R}$ is called a *norm* if the following three axioms hold:

N1. $\|x\| \geq 0$ for all $x \in E$, and $\|x\| = 0$ if and only if $x = 0$.
N2. $\|ax\| = |a|\|x\|$ for all $a \in \mathbb{K}$, $x \in E$.
N3. (Triangle Inequality). $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in E$.

A pair $(E, \|..\|)$ consisting of a vector space and a norm on it is called a *normed space*.

Note. *If $(E, \|..\|)$ is a normed space, $d(x, y) := \|x - y\|$ defines a metric whose topology makes E into a topological vector space.*

Definition (Banach Space). A normed vector space is called a *Banach space* if it is complete, i.e. if every Cauchy sequence converges.

Hilbert and Banach spaces are, *in particular*, examples of topological vector spaces, but they have more structure than that: The scalar product $\langle .., .. \rangle$ or the norm $\|..\|$ obviously cannot be recovered from the topology. Already for finite $n \geq 2$, a vector space V of dimension n can be endowed with many different norms which—in contrast with scalar products—cannot be

obtained from one another by linear isomorphisms of the space into itself. Of course, all these norms define the same (i.e. the "usual") topology on V. Now, in *infinite* dimensions, even if one is only interested in the topological vector space structure (as often happens in functional analysis), Banach spaces define a very rich class of which it is difficult and perhaps impossible to get a complete overview.

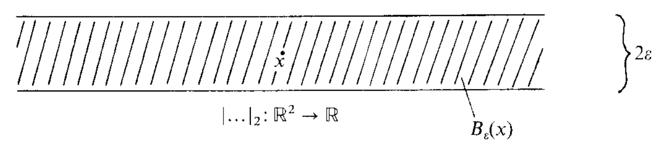
§5. Fréchet Spaces

Definition (Seminorm). Let E be a $\mathbb{K}$-vector space. A map $|..|: E \to \mathbb{R}$ is called a seminorm if the following hold:

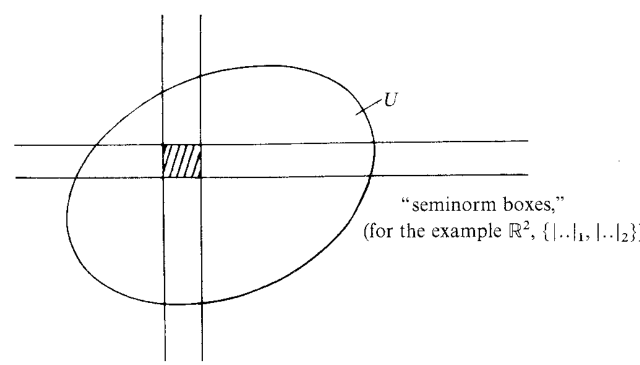
SN1. $|x| \geq 0$ for all $x \in E$.
N2. $|ax| = |a||x|$, } as for norms.
N3. Triangle inequality, }

For example, $|..|_i: \mathbb{R}^n \to \mathbb{R}$, $x \mapsto |x_i|$ is a seminorm on $\mathbb{R}^n$.

We can talk about "open balls" for seminorms as well as for norms, and we will denote them by $B_\varepsilon(x) := \{y \in E \mid |x - y| < \varepsilon\}$; but in general there isn't anything "round" about them anymore.



Definition. Let E be a vector space and $\{|..|_\lambda\}_{\lambda \in \Lambda}$ a family of seminorms on E. A subset $U \subset E$ is called open in the topology generated by the family of seminorms if every point of U belongs to a finite intersection of seminorm open balls which is contained in U; in other words, for every $x \in U$ there are $\lambda_1, \ldots, \lambda_r \in \Lambda$ and an $\varepsilon > 0$ such that $B_\varepsilon^{(\lambda_1)}(x) \cap \cdots \cap B_\varepsilon^{(\lambda_r)}(x) \subset U$.



In the terminology of I, §4 these open balls of the seminorms $|..|_\lambda$, $\lambda \in \Lambda$ form a subbasis of, or generate, the topology.

Note. *With the topology given by the family of seminorms* $\{|..|_\lambda\}_{\lambda\in\Lambda}$, *E becomes a topological vector space, which moreover is Hausdorff if and only if* 0 *is the only vector for which all seminorms* $|..|_\lambda$ *are zero.*

Definition (Pre-Fréchet Space). A Hausdorff topological vector space whose topology can be defined by an at most countable family of seminorms is called a *pre-Fréchet space.*

Fréchet spaces will be the "complete" pre-Fréchet spaces. To be sure, completeness is a metric motion, but there is an obvious topological version of it for topological vector spaces:

Definition (Complete Topological Vector Spaces). A sequence $(x_n)_{n\geq 1}$ in a topological vector space is called a *Cauchy sequence* if for every neighborhood U of 0 there is an n_0 such that $x_n - x_m \in U$ for all $n, m \geq n_0$. If every Cauchy sequence converges, the space is called *(sequentially) complete.*

In normed spaces this concept of completeness is of course equivalent to the old one, obtained from the metric given by the norm.

Definition (Fréchet Space). A Fréchet space is a complete pre-Fréchet space.

Notice that pre-Fréchet spaces are always metrizable: If the topology is given by a sequence of seminorms $|..|_{n, n\geq 1}$, then

$$d(x, y) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x - y|_n}{1 + |x - y|_n}$$

defines a metric which generates the same topology and for which the Cauchy sequences are the same.

§6. Locally Convex Topological Vector Spaces

Finally, let us define locally convex spaces, which are the most general class of topological vector spaces for which there exists a theory with decent theorems.

Definition. A topological vector space is called locally convex if every neighborhood of 0 contains a convex neighborhood of 0.

We mention the following facts to illustrate to what extent these spaces are more general than the preceding ones (no proof here; cf. [13] §18): A topological vector space is locally convex if and only if its topology can be given by a family of seminorms; and a locally convex topological vector space is a pre-Fréchet space if and only if it is metrizable.

§7. A Couple of Examples

Example 1. We consider the Lebesgue-integrable real functions f on $[-\pi, \pi]$ which satisfy

$$\int_{-\pi}^{\pi} f(x)^2 \, dx < \infty.$$

Two such functions will be called equivalent if they coincide outside a set of measure zero. The equivalence classes are called, somewhat loosely, square-integrable functions. Let H be the set of such functions. H has a canonical real vector space structure and can be made into a Hilbert space using, for instance, the following inner product:

$$\langle f, g\rangle := \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x) \, dx.$$

The trigonometric functions $e_k := \cos kx$, $e_{-k} := \sin kx$, $k \geq 1$, form, together with $e_0 := \sqrt{2}/2$, a Hilbert basis $\{e_n\}_{n\in\mathbb{Z}}$ for H, and the representation of elements $f \in H$ as $f = \sum_{n\in\mathbb{Z}} \langle f, e_n\rangle e_n$ is exactly the Fourier series of f.

Example 2. Let X be a topological space, $C(X)$ the vector space of bounded continuous functions on X, and $\|f\| := \sup_{x\in X} |f(x)|$. Then $(C(X), \|..\|)$ is a Banach space.

Example 3. Let $X \subset \mathbb{C}$ be a domain and $\mathcal{O}(X)$ the vector space of holomorphic functions on X, endowed with the topology given by the family

$$\{|..|_K\}_{K\subset X \text{ is compact}}$$

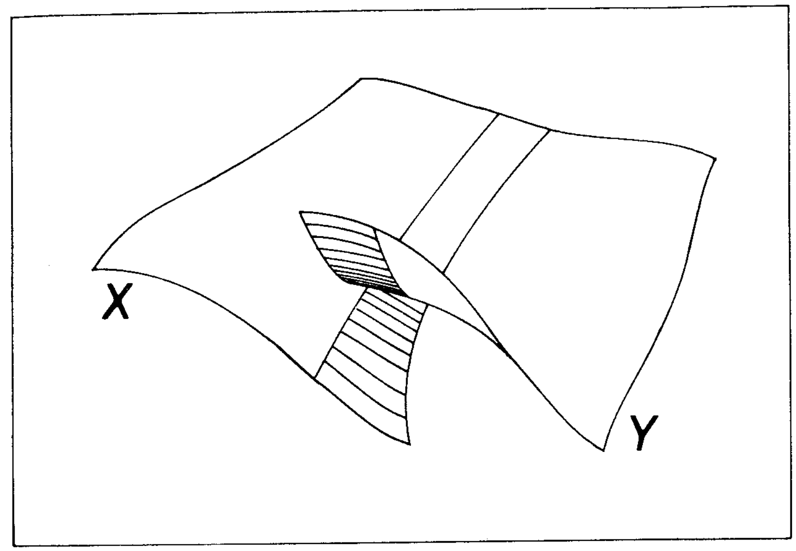
of seminorms $|f|_K := \sup_{z\in K} |f(z)|$ (topology of "compact convergence"). Then $\mathcal{O}(X)$ is a Fréchet space (we just have to consider a countable collection of K_n which "exhaust" X; completeness follows from the Weierstrass convergence theorem ...).

These are three out of a great number of "function spaces" which effectively come up in analysis. As mere vector spaces they did not have to be invented, they just are there and one can't miss them. And that the linear differential and integral operators behave as linear maps $L\colon E_1 \to E_2$ between function spaces also follows immediately from the nature of things. But mere linear algebra will lead us only to trivialities here; to understand the properties of these operators, we must study their continuity behavior under different topologies, and exploit our knowledge about the structure of abstract topological vector spaces. And while point-set topology, in whose praise I'm saying all this, does not exactly represent the cutting edge of research in the area of partial differential equations, it is nevertheless an indispensable instrument in it, to the point of being taken for granted.

I haven't yet given any examples of locally convex but non-metrizable, and hence non-pre-Fréchet, topological vector spaces. Well, such spaces also come up in a completely natural way in function analysis. For instance, it is sometimes necessary to consider the "weak topology" on a given topological vector space, that is, the coarsest topology for which all the old continuous linear maps $E \to \mathbb{R}$ (the "linear functionals") remain continuous, or in other words, the topology generated by $\{f^{-1}(U) \mid U \subset \mathbb{R}$ is open, $f: E \to \mathbb{R}$ is linear and continuous$\}$. With this topology E is still a topological vector space, but much more complicated than before. Even if we start with something as simple as an infinite-dimensional Hilbert space, we end up with a locally convex, Hausdorff, but non-metrizable topological vector space (cf. [4], p. 76).

CHAPTER III

The Quotient Topology



§1. The Notion of a Quotient Space

Notation. If X is a set and $\sim$ an equivalence relation on X, then $X/\sim$ will denote the set of equivalence classes, $[x] \in X/\sim$ the equivalence class of $x \in X$, and $\pi: X \to X/\sim$ the canonical projection, so that $\pi(x) := [x]$.

Definition (Quotient Space). Let X be a topological space and $\sim$ an equivalence relation on X. A set $U \subset X/\sim$ is called *open in the quotient topology* if $\pi^{-1}(U)$ is open in X. $X/\sim$, endowed with the topology thus defined, is called the *quotient of* X *by* $\sim$.

Note. *The quotient topology is obviously the finest topology on* $X/\sim$ *such that* π *is a continuous map.*

Just as we have, for the notions of subspace, disjoint union and product, a simple mental image on which we can base our intuition in the beginning, I would like to suggest a mental image for quotient spaces as well. In order to depict an equivalence relation, the best thing is to imagine the equivalence