§1. SMOOTH MANIFOLDS AND SMOOTH MAPS

FIRST let us explain some of our terms. R^k denotes the k-dimensional euclidean space; thus a point $x \in R^k$ is an k-tuple $x = (x_1, \ldots, x_k)$ of real numbers.

Let $U \subset R^k$ and $V \subset R^l$ be open sets. A mapping f from U to V (written $f : U \to V$) is called *smooth* if all of the partial derivatives $\partial^n f/\partial x_{i_1} \cdots \partial x_{i_n}$ exist and are continuous.

More generally let $X \subset R^k$ and $Y \subset R^l$ be arbitrary subsets of euclidean spaces. A map $f : X \to Y$ is called *smooth* if for each $x \in X$ there exist an open set $U \subset R^k$ containing x and a smooth mapping $F : U \to R^l$ that coincides with f throughout $U \cap X$.

If $f : X \to Y$ and $g : Y \to Z$ are smooth, note that the composition $g \circ f : X \to Z$ is also smooth. The identity map of any set X is automatically smooth.

DEFINITION. A map $f : X \to Y$ is called a *diffeomorphism* if f carries X homeomorphically onto Y and if both f and f^{-1} are smooth.

We can now indicate roughly what *differential topology* is about by saying that it studies those properties of a set $X \subset R^k$ which are invariant under diffeomorphism.

We do not, however, want to look at completely arbitrary sets X. The following definition singles out a particularly attractive and useful class.

DEFINITION. A subset $M \subset R^k$ is called a *smooth manifold* of *dimension* m if each $x \in M$ has a neighborhood $W \cap M$ that is diffeomorphic to an open subset U of the euclidean space R^m.

Any particular diffeomorphism $g : U \to W \cap M$ is called a *parametrization* of the region $W \cap M$. (The inverse diffeomorphism $W \cap M \to U$ is called a system of *coordinates* on $W \cap M$.)

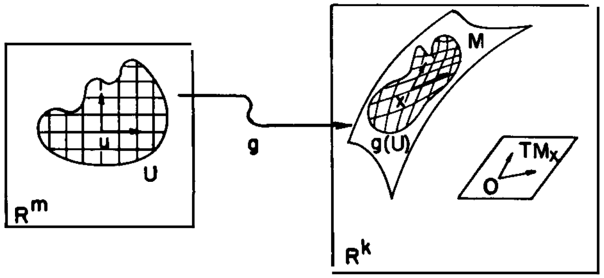


Figure 1. Parametrization of a region in M

Sometimes we will need to look at manifolds of dimension zero. By definition, M is a manifold of dimension zero if each $x \in M$ has a neighborhood $W \cap M$ consisting of x alone.

Examples. The unit sphere S^2, consisting of all $(x, y, z) \in R^3$ with $x^2 + y^2 + z^2 = 1$ is a smooth manifold of dimension 2. In fact the diffeomorphism

$$(x, y) \to (x, y, \sqrt{1 - x^2 - y^2}),$$

for $x^2 + y^2 < 1$, parametrizes the region $z > 0$ of S^2. By interchanging the roles of x, y, z, and changing the signs of the variables, we obtain similar parametrizations of the regions $x > 0$, $y > 0$, $x < 0$, $y < 0$, and $z < 0$. Since these cover S^2, it follows that S^2 is a smooth manifold.

More generally the sphere $S^{n-1} \subset R^n$ consisting of all $(x_1, \cdots, x_n)$ with $\sum x_i^2 = 1$ is a smooth manifold of dimension $n - 1$. For example $S^0 \subset R^1$ is a manifold consisting of just two points.

A somewhat wilder example of a smooth manifold is given by the set of all $(x, y) \in R^2$ with $x \neq 0$ and $y = \sin(1/x)$.

TANGENT SPACES AND DERIVATIVES

To define the notion of *derivative* df_x for a smooth map $f : M \to N$ of smooth manifolds, we first associate with each $x \in M \subset R^k$ a linear subspace $TM_x \subset R^k$ of dimension m called the *tangent space* of M at x. Then df_x will be a linear mapping from TM_x to TN_y, where $y = f(x)$. Elements of the vector space TM_x are called *tangent vectors* to M at x.

Intuitively one thinks of the m-dimensional hyperplane in R^k which best approximates M near x; then TM_x is the hyperplane through the

origin that is parallel to this. (Compare Figures **1** and **2.**) Similarly one thinks of the nonhomogeneous linear mapping from the tangent hyperplane at x to the tangent hyperplane at y which best approximates f. Translating both hyperplanes to the origin, one obtains df_x.

Before giving the actual definition, we must study the special case of mappings between open sets. For any open set $U \subset R^k$ the *tangent space* TU_x is defined to be the entire vector space R^k. For any smooth map $f: U \to V$ the *derivative*

$$df_x : R^k \to R^l$$

is defined by the formula

$$df_x(h) = \lim_{t \to 0} (f(x + th) - f(x))/t$$

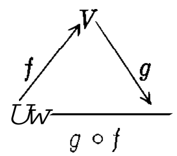
for $x \in U$, $h \in R^k$. Clearly $df_x(h)$ is a linear function of h. (In fact df_x is just that linear mapping which corresponds to the $l \times k$ matrix $(\partial f_i/\partial x_j)_x$ of first partial derivatives, evaluated at x.)

Here are two fundamental properties of the derivative operation:

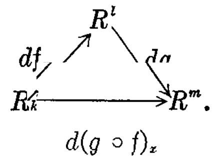
1 (Chain rule). *If $f : U \to V$ and $g : V \to W$ are smooth maps, with $f(x) = y$, then*

$$d(g \circ f)_x = dg_y \circ df_x.$$

In other words, to every commutative triangle



of smooth maps between open subsets of R^k, R^l, R^m there corresponds a commutative triangle of linear maps



2. *If I is the identity map of U, then dI_x is the identity map of R^k. More generally, if $U \subset U'$ are open sets and*

$$i : U \to U'$$

smooth map

$$f : M \to N$$

with $f(x) = y$. The *derivative*

$$df_x : TM_x \to TN_y$$

is defined as follows. Since f is smooth there exist an open set W containing x and a smooth map

$$F : W \to R^l$$

that coincides with f on $W \cap M$. *Define $df_x(v)$ to be equal to $dF_x(v)$ for all $v \in TM_x$.*

To justify this definition we must prove that $dF_x(v)$ belongs to TN_y and that it does not depend on the particular choice of F.

Choose parametrizations

$$g : U \to M \subset R^k \quad \text{and} \quad h : V \to N \subset R^l$$

for neighborhoods $g(U)$ of x and $h(V)$ of y. Replacing U by a smaller set if necessary, we may assume that $g(U) \subset W$ and that f maps $g(U)$ into $h(V)$. It follows that

$$h^{-1} \circ f \circ g : U \to V$$

is a well-defined smooth mapping.

Consider the commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{\quad F \quad} & R^l \\ g \uparrow & & \uparrow h \\ U & \xrightarrow{h^{-1} \circ f \circ g} & V \end{array}$$

of smooth mappings between open sets. Taking derivatives, we obtain a commutative diagram of linear mappings

$$\begin{array}{ccc} R^k & \xrightarrow{\quad dF_x \quad} & R^l \\ dg_u \uparrow & & \uparrow dh_v \\ R^m & \xrightarrow{d(h^{-1} \circ f \circ g)_u} & R^n \end{array}$$

where $u = g^{-1}(x)$, $v = h^{-1}(y)$.

It follows immediately that dF_x carries $TM_x = \text{Image}\ (dg_u)$ into $TN_y = \text{Image}\ (dh_v)$. Furthermore the resulting map df_x does not depend on the particular choice of F, for we can obtain the same linear transformation by going around the bottom of the diagram. That is:

$$df_x = dh_v \circ d(h^{-1} \circ f \circ g)_u \circ (dg_u)^{-1}.$$

This completes the proof that

$$df_x : TM_x \to TN_y$$

is a well-defined linear mapping.

As before, the derivative operation has two fundamental properties:

1. (Chain rule). *If $f : M \to N$ and $g : N \to P$ are smooth, with $f(x) = y$, then*

$$d(g \circ f)_x = dg_y \circ df_x.$$

2. *If I is the identity map of M, then dI_x is the identity map of TM_x. More generally, if $M \subset N$ with inclusion map i, then $TM_x \subset TN_x$ with inclusion map di_x.* (Compare Figure **2.**)

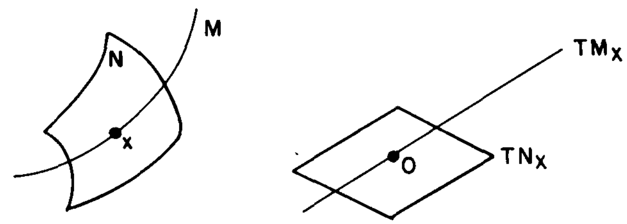


Figure 2. The tangent space of a submanifold

The proofs are straightforward.

As before, these two properties lead to the following:

ASSERTION. *If $f : M \to N$ is a diffeomorphism, then $df_x : TM_x \to TN_y$ is an isomorphism of vector spaces. In particular the dimension of M must be equal to the dimension of N.*

REGULAR VALUES

Let $f : M \to N$ be a smooth map between manifolds of the same dimension.* We say that $x \in M$ is a *regular point* of f if the derivative

* This restriction will be removed in §2.

df_x is nonsingular. In this case it follows from the inverse function theorem that f maps a neighborhood of x in M diffeomorphically onto an open set in N. The point $y \in N$ is called a *regular value* if $f^{-1}(y)$ contains only regular points.

If df_x is singular, then x is called a *critical point* of f, and the image $f(x)$ is called a *critical value.* Thus each $y \in N$ is either a critical value or a regular value according as $f^{-1}(y)$ does or does not contain a critical point.

Observe that if M is compact and $y \in N$ is a regular value, then $f^{-1}(y)$ is a finite set (possibly empty). For $f^{-1}(y)$ is in any case compact, being a closed subset of the compact space M; and $f^{-1}(y)$ is discrete, since f is one-one in a neighborhood of each $x \in f^{-1}(y)$.

For a smooth $f : M \to N$, with M compact, and a regular value $y \in N$, we *define $\#f^{-1}(y)$ to be the number of points in $f^{-1}(y)$.* The first observation to be made about $\#f^{-1}(y)$ is that it is locally constant as a function of y (where y ranges only through regular values!). I.e., *there is a neighborhood $V \subset N$ of y such that $\#f^{-1}(y') = \#f^{-1}(y)$ for any $y' \in V$.* [Let $x_1, \cdots, x_k$ be the points of $f^{-1}(y)$, and choose pairwise disjoint neighborhoods $U_1, \cdots, U_k$ of these which are mapped diffeomorphically onto neighborhoods $V_1, \cdots, V_k$ in N. We may then take

$$V = V_1 \cap V_2 \cap \cdots \cap V_k - f(M - U_1 - \cdots - U_k).]$$

THE FUNDAMENTAL THEOREM OF ALGEBRA

As an application of these notions, we prove the fundamental theorem of algebra: *every nonconstant complex polynomial $P(z)$ must have a zero.*

For the proof it is first necessary to pass from the plane of complex numbers to a compact manifold. Consider the unit sphere $S^2 \subset R^3$ and the stereographic projection

$$h_+ : S^2 - \{(0, 0, 1)\} \to R^2 \times 0 \subset R^3$$

from the "north pole" $(0, 0, 1)$ of S^2. (See Figure 3.) We will identify $R^2 \times 0$ with the plane of complex numbers. The polynomial map P from $R^2 \times 0$ itself corresponds to a map f from S^2 to itself; where

$$f(x) = h_+^{-1} P h_+(x) \quad \text{for} \quad x \neq (0, 0, 1)$$

$$f(0, 0, 1) = (0, 0, 1).$$

It is well known that this resulting map f is smooth, even in a neighbor-

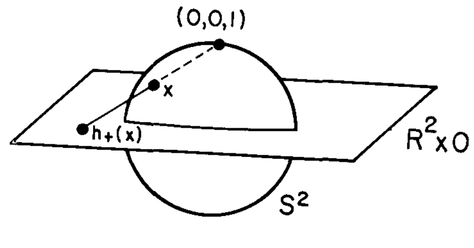


Figure 3. Stereographic projection

hood of the north pole. To see this we introduce the stereographic projection h_- from the south pole $(0, 0, -1)$ and set

$$Q(z) = h_- f h_-^{-1}(z).$$

Note, by elementary geometry, that

$$h_+ h_-^{-1}(z) = z/|z|^2 = 1/\bar{z}.$$

Now if $P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$, with $a_0 \neq 0$, then a short computation shows that

$$Q(z) = z^n/(\bar{a}_0 + \bar{a}_1 z + \cdots + \bar{a}_n z^n).$$

Thus Q is smooth in a neighborhood of 0, and it follows that $f = h_-^{-1} Q h_-$ is smooth in a neighborhood of $(0, 0, 1)$.

Next observe that f has only a finite number of critical points; for P fails to be a local diffeomorphism only at the zeros of the derivative polynomial $P'(z) = \sum a_{n-j}\, j z^{j-1}$, and there are only finitely many zeros since P' is not identically zero. The set of regular values of f, being a sphere with finitely many points removed, is therefore connected. Hence the locally constant function $\#f^{-1}(y)$ must actually be constant on this set. Since $\#f^{-1}(y)$ can't be zero everywhere, we conclude that it is zero nowhere. Thus f is an onto mapping, and the polynomial P must have a zero.