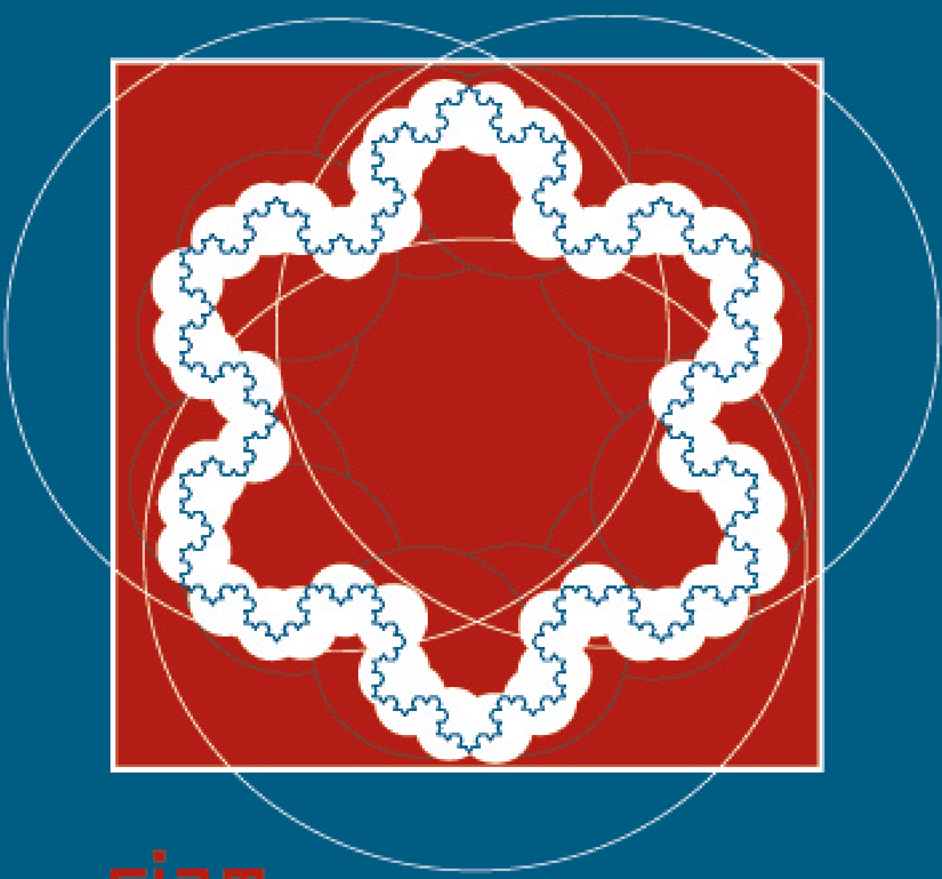
Differential Dynamical Systems

James D. Meiss



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James D. Meiss
University of Colorado
Boulder, Colorado

Society for Industrial and Applied Mathematics
Philadelphia

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Contents

Preface

On one level, this text can be viewed as suitable for a traditional course on ordinary differential equations (ODEs). Since differential equations are the basis for models of any physical systems that exhibit smooth change, students in all areas of the mathematical sciences and engineering require the tools to understand the methods for solving these equations. It is traditional for this exposure to start during the second year of training in calculus, where the basic methods of solving one- and two-dimensional (primarily linear) ODEs are studied. The typical reader of this text will have had such a course, as well as an introduction to analysis where the theoretical foundations (the ε's and δ's) of calculus are elucidated. The material for this text has been developed over a decade in a course given to upper-division undergraduates and beginning graduate students in applied mathematics, engineering, and physics at the University of Colorado. In a one-semester course, I typically cover most of the material in Chapters 1–6 and add a selection of sections from later chapters.

There are a number of classic texts for a traditional differential equations course, for example (Coddington and Levinson 1955; Hirsch and Smale 1974; Hartman 2002). Such courses usually begin with a study of linear systems; we begin there as well in Chapter 2. Matrix algebra is fundamental to this treatment, so we give a brief discussion of eigenvector methods and an extensive treatment of the matrix exponential. The next stage in the traditional course is to provide a foundation for the study of nonlinear differential equations by showing that, under certain conditions, these equations have solutions (existence) and that there is only one solution that satisfies a given initial condition (uniqueness). The theoretical underpinning of this result, as well as many other results in applied mathematics, is the majestic contraction mapping theorem. Chapter 3 provides a self-contained introduction to the analytic foundations needed to understand this theorem. Once this tool is concretely understood, students see that many proofs quickly yield to its power. It is possible to omit §§3.3–3.5, as most of the material is not heavily used in later chapters, although at least passing acquaintance with Theorem 3.10 and Lemma 3.13 (Grönwall) is to be encouraged.

However, this text does not aim to cover only the material in such a traditional ODE course; rather, it aspires to serve as an introduction to the more modern theory of dynamical systems. The emphasis is on obtaining a *qualitative* understanding of the properties of *differential* dynamical systems, namely, those evolution rules that describe smooth evolution in time.[1] The primary concept of this study, the *flow*, is introduced in Chapter 4. The

[1] This is not to say that the dynamical systems that we study are always *differentiable*—vector fields need not be smooth.

qualitative theory is often concerned with questions of shape and asymptotic behavior that lead us to use topological notions such as conjugacy in the classification of dynamics.

The classification of dynamical behavior begins with the simplest orbits, equilibria and periodic orbits. As Henri Poincaré noted in his classic *New Methods in Celestial Mechanics*, (1892, Vol. 1, §36),

> *what renders these periodic solutions so precious to us is that they are, so to speak, the only breach through which we may attempt to penetrate an area hitherto deemed inaccessible.*

Only in the demonstration that dynamics in the neighborhood of some of these orbits is conjugate to their linearization is it seen that the predisposition of applied scientists to concentrate on linear systems has any value whatsoever.

The local classification of equilibria leads to the theory of invariant manifolds in Chapter 5. The stable and unstable manifolds, proved to exist for a hyperbolic saddle, give rise to one prominent mechanism for chaos—heteroclinic intersection. The center manifold theorem is also important preparation for the treatment of bifurcations in Chapter 8.

As mathematicians, allow yourselves to become entranced by the exceptions to the validity of linearization, namely, with those orbits that are nonhyperbolic. It is in the study of these exceptions that we find the most beautiful dynamics—even in the case of the phase plane, to which we return in Chapter 6. The first three sections of this chapter are fundamental; §§6.4–6.8 can be omitted in favor of later chapters. As we see in Chapter 8, the exceptional cases form the organizing centers for the behavior of systems undergoing changing parameters. A qualitative change in behavior under a small change of parameters is called a bifurcation. A complete exegesis of theory of bifurcations requires a full text on its own, and there are many excellent texts appropriate for a more advanced class (Guckenheimer and Holmes 1983; Golubitsky and Schaeffer 1985; Kuznetsov 1995). We introduce the reader to the basic ideas of normal forms and treat codimension-one and -two bifurcations.

Perhaps the most exciting recent developments in dynamical systems are those that show that even simple systems can behave in complicated ways, namely, the phenomena of *chaos*. In Chapter 7, we introduce the reader to the concepts necessary for understanding chaos: Lyapunov exponents, transitivity, fractals, etc. We also give an extensive discussion of Melnikov's method for the onset of chaos in Chapter 8. A more advanced treatment of chaotic dynamics requires a discussion of discrete dynamics (mappings) and can be found in texts such as (Katok and Hasselblatt 1999; Robinson 1999; Wiggins 2003).

The final chapter treats the subject closest to this author's heart: Hamiltonian dynamics. Since the basic models of physics all have a Hamiltonian (or Lagrangian) formulation, it is worthwhile to become familiar with them. While a traditional physics text treats these on a concrete level, this book provides an introduction to some of the geometrical aspects of Hamiltonian dynamics, including a discussion of their variational foundation, spectral properties, the KAM theorem, and transition to chaos. Again, there are several advanced texts that go much further, for example (Arnold 1978; Lichtenberg and Lieberman 1992; Meyer and Hall 1992).

While the proofs of many of the classical theorems are included, this text is not just an abstract treatment of ODEs but an attempt to place the theory in the context of its many applications to physics, biology, chemistry, and engineering. Examples in such areas as population modeling, fluid convection, electronics, and mechanics are discussed throughout the text, and especially in Chapter 1. The exercises introduce the reader to many more. Furthermore, to develop a geometrical understanding of dynamics, each student must experiment; we provide some examples of simple codes written in Maple, Mathematica, and MATLAB in the appendix, and we use the exercises to encourage the student to explore further. There are several texts that focus completely on using one or more of tools like these to explore dynamics (Lynch 2001; Baumann 2004).

I hope that this book conveys a bit of my amazement with the beauty and utility of this field. Dynamical systems is the perfect combination of analysis, geometry, and physical intuition. Central questions in dynamics have been formulated for centuries, and although some have been solved in the past few years, many await solution by the next generation.

> *It is far better to foresee even without certainty than not to foresee at all.* (Henri Poincaré, *The Foundations of Science*)

James Meiss
Boulder, Colorado
March 2007

Chapter 4

Dynamical Systems

> *Science, as well as history, has its past to show—a past indeed, much larger; but its immensity is dynamic, not divine.* (James Martineau)

So far, our approach to the study of dynamics has been completely traditional: we concentrated on some simple, solvable systems—especially linear systems—and we proved that more general, nonlinear systems actually have solutions. By contrast, the theory of "dynamical systems" is more concerned with qualitative properties. In this chapter we will seek to develop a classification of the qualitative properties of dynamics and to understand asymptotic behavior—what happens as $t \to \infty$. The first part of this study concerns the trajectories of a dynamical system in a local neighborhood. The goals are to classify equilibria by their stability, invariant manifolds, and topological type. This information will be used in later chapters to understand bifurcations and global dynamics.

4.1 Definitions

> *Behold the rule we follow, and the only one we can follow: when a phenomenon appears to us as the cause of another, we regard it as anterior. It is therefore by cause that we define time.* (Henri Poincaré, 1914)

According to the *Encyclopedia Britannica*, dynamics is the "branch of physical science that is concerned with the motion of material objects in relation to the physical factors that affect them: force, mass, momentum, energy." Since Newton showed that mechanical systems are governed by differential equations, these do indeed provide good examples of dynamics. However, a more general definition is

> ▹ *dynamical system*: An *evolution rule* that defines a trajectory as a function of a single parameter (*time*) on a set of *states* (the *phase space*) is a dynamical system.

Dynamical systems are therefore categorized according to properties of their phase space, of their evolution rule, and of time itself. In this book, we consider systems with a continuous phase space, M, that is typically $\mathbb{R}^n$ or a more general space called a "manifold" such as the cylinder or torus.[21] Systems with a discrete phase space include the heads–tails model of a coin toss and "cellular automata" (Wolfram 1983). We will also primarily study systems with a continuous time variable, $t \in \mathbb{R}$. Systems with a discrete time variable are called "mappings" (Alligood, Sauer, and Yorke 1997; Devaney 1986).

The evolution rule can be deterministic or stochastic. A system is *deterministic* if for each state in the phase space there is a unique consequent, i.e., the evolution rule is a function taking a given state to a unique, subsequent state. Systems that are nondeterministic are called *stochastic*: a standard example is the idealized coin toss. For this case, the phase space is finite, consisting of the two states, heads and tails, and time is discrete taking the values at which the coin is examined. The evolution rule states that a head or a tail is equally likely at the next toss, independent of the current state of the coin.

When the evolution rule is deterministic, then for each time, t, it is a mapping from the phase space to the phase space,

$$\varphi_t : M \to M, \tag{4.1}$$

so that $x(t) = \varphi_t(x_o)$ denotes the position of the system at time t that started at x_o. Here we assume that t takes values in some allowed range and that the initial value of time is zero, so that $\varphi_o(x_o) = x_o$.

Every dynamical system has *orbits* or *trajectories*; namely, the sequence of states that follow from or lead to a given initial state. The forward orbit is the set of subsequent states

$$\Gamma_x^+ \equiv \{\varphi_t(x) : t \geq 0\}. \tag{4.2}$$

Similarly, the *preorbit* is the set of sequences of states that lead, according to the evolution rule, to the initial state. When the function φ_t is one to one, then the preorbit is simply the set $\{\varphi_t(x) : t \leq 0\}$; otherwise, it is possible that several prior points could lead to the same x. Finally, the full *orbit* of a point x, Γ_x, is simply the union of the forward and preorbits of x.

The simplest orbit is an *equilibrium*, where the orbit is a single point: $\Gamma_x = \{x\}$. A *periodic orbit*, γ, is a closed loop; it can be viewed as an embedding of the circle $\mathbb{S}^1$ into the phase space, $\gamma : \mathbb{S}^1 \to \mathbb{R}^n$. Note that for each x on a periodic orbit, there is a time T such that the point returns to itself:

$$\varphi_T(x) = x. \tag{4.3}$$

More generally orbits can be quasiperiodic, aperiodic, or chaotic; we will discuss these in later sections.

An orbit is a special case of an

▷ *invariant set*: A set Λ is invariant under a rule φ_t if $\varphi_t(\Lambda) = \Lambda$ for all t; that is, for each $x \in \Lambda$, $\varphi_t(x) \in \Lambda$ for any t.

Thus for each point x in an invariant set Λ, the entire orbit of x must be in Λ as well. Just as we define a forward orbit, we can also define a

▷ *forward invariant set*: A set Λ is forward invariant if $\varphi_t(\Lambda) \subset \Lambda$ for all $t > 0$.

[21] For our purposes, it is sufficient to think of a manifold simply as a smooth, multidimensional surface embedded in $\mathbb{R}^n$; see §5.5. More formal definitions are given in courses on differential geometry.

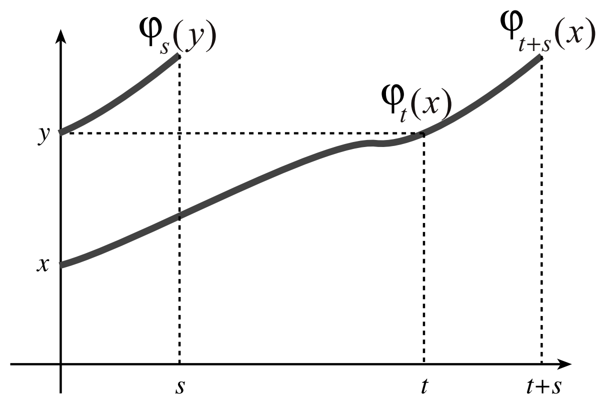

Figure 4.1. *Illustration of the group property of a flow,* $\varphi_s(y) = \varphi_s(\varphi_t(x)) = \varphi_{t+s}(x)$.

4.2 Flows

In §3.4 the solution of the initial value problem (3.26) with initial condition y was denoted by $u(t; y)$, and it was shown that u is a C^1 function of both t and y when the vector field is C^1. In this section, we will let

$$u(t; y) \to \varphi_t(y),$$

as in (4.1), so that the evolution rule is now thought of as a map from the phase space to itself that is parameterized by time. To emphasize this change of point-of-view, we define a class of evolution rules without reference to ordinary differential equations (ODEs):

> ▷ *flow*: Suppose the phase space for a dynamical system is a manifold M. A *complete flow* $\varphi_t(x)$ is a one-parameter, differentiable mapping $\varphi : \mathbb{R} \times M \to M$, such that
>
> (a) $\varphi_0(x) = x$, and
>
> (b) for all t and $s \in \mathbb{R}$,
> $$\varphi_t \circ \varphi_s = \varphi_{t+s}, \tag{4.4}$$
> where the composition symbol, $\circ$, means $\varphi_t \circ \varphi_s(x) \equiv \varphi_t(\varphi_s(x))$.

For each fixed x, $\varphi_t(x)$ defines a curve in M as t varies over $\mathbb{R}$—the orbit (4.2). Property (b) is known as the *group property*, since it implies that under the operation of composition, the family of maps $\{\varphi_t : t \in \mathbb{R}\}$ is an additive group (see Figure 4.1). For example, the group property for $s = -t$ implies $\varphi_t \circ \varphi_{-t} = \varphi_0 = id$ (here id is the "identity" function, $id(x) = x$), hence φ_t is an invertible function of x for each t, and moreover

$$(\varphi_t)^{-1} = \varphi_{-t}.$$

Consequently, for each t the flow φ_t is one-to-one and onto map on M: it is a *bijection*. The group property also implies that two distinct trajectories cannot cross: if two trajectories ever touch, say, at a point $y = \varphi_t(x) = \varphi_s(z)$, then the group property implies that $\varphi_{t+r}(x) = \varphi_{s+r}(z)$ for all $r \in \mathbb{R}$, and the trajectories coincide.

Example: If φ is a flow and γ is a periodic orbit, then the group property and (4.3) imply that

$$\varphi_{T+s}(x) = \varphi_s(x),$$

and so $\varphi_{2T}(x) = x$ and indeed $\varphi_{kT}(x) = x$ for any integer k. If T is the minimum positive value for which $\varphi_T(x) = x$, it is called the *period* of γ. It is also easy to see from the group property that if y is any other point on γ, then it has the same period as x: $\varphi_T(y) = y$. ∎

A flow is *complete* when it is defined for all t, so that the group property applies for all time. Usually, when we use the term "flow" without any qualification we mean a complete flow. Note that the group property implies that $x(t) = \varphi_{t-s}(\varphi_s(x_o)) = \varphi_{t-s}(x(s))$ for any time s along the trajectory. Therefore, $x(s)$ can also be viewed as the "initial condition" for the trajectory $x(t)$, but one that is defined at the time s.

Since a flow is differentiable, it has an associated ODE, or more precisely a

▷ *vector field*: A vector field is a function $f : M \to \mathbb{R}^n$ that defines a vector $v = f(x)$ at each point x in the phase space M.

The vector field associated with a flow is defined by

$$f(x) = \left.\frac{d}{dt}\varphi_t(x)\right|_{t=0}. \tag{4.5}$$

This vector field is interesting because the flow is a solution of the differential equation $\dot{x} = f(x)$, as we show next.

Lemma 4.1. *If $\varphi_t(x)$ is a flow, then it is a solution of the initial value problem*

$$\frac{d}{dt}\varphi_t(x_o) = f(\varphi_t(x_o)), \quad \varphi_o(x_o) = x_o,$$

for the vector field defined in (4.5).

Proof. Let $x(t) = \varphi_t(x_o)$. Differentiating and using the group property yields

$$\frac{dx}{dt} = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\varphi_{t+\varepsilon}(x_o) - \varphi_t(x_o)\right] = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\varphi_\varepsilon(x(t)) - \varphi_o(x(t))\right] = f(x(t)).$$

Therefore, the flow is the solution of the differential equation $\dot{x} = f(x)$. □

When the flow is complete, the solutions to this differential equation exist for all time: their maximal interval of existence is $(-\infty, \infty)$.

Example: The function $\varphi_t(x) = xe^{\lambda t}$ is a smooth map $\varphi : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, and can be seen to satisfy the flow properties (a) and (b). Differentiation gives $\frac{d}{dt}\varphi_t(x) = \lambda\varphi_t(x)$, so that the vector field associated with φ_t is simply $f(x) = \lambda x$. Of course, φ_t is the general solution of the ODE $\dot{x} = \lambda x$. ∎

Example: Consider the function $\varphi_t : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\varphi_t(x) = \begin{pmatrix} \varphi_{1t}(x) \\ \varphi_{2t}(x) \end{pmatrix} = \begin{pmatrix} x_1 e^{-t} \\ x_2 e^{x_1(e^{-t}-1)} \end{pmatrix}.$$

This function is clearly defined for all $(x_1, x_2) \in \mathbb{R}^2$ and $t \in \mathbb{R}$, and it is C^1 on this domain. To see that it satisfies the flow properties note first that $\varphi_0(x) = x$ and that

$$\varphi_t(\varphi_s(x)) = \begin{pmatrix} \varphi_{1s}(x)e^{-t} \\ \varphi_{2s}(x)e^{\varphi_{1s}(x)(e^{-t}-1)} \end{pmatrix} = \begin{pmatrix} x_1 e^{-(s+t)} \\ x_2 e^{x_1(e^{-s}-1)} e^{x_1 e^{-s}(e^{-t}-1)} \end{pmatrix} = \varphi_{s+t}(x).$$

Thus $\varphi_t(x)$ is a flow. The vector field (4.5) associated with this flow is given by differentiation:

$$\left.\frac{d}{dt}\varphi_t(x)\right|_{t=0} = \left.\begin{pmatrix} -x_1 e^{-t} \\ -x_2 x_1 e^{-t} e^{x_1(e^{-t}-1)} \end{pmatrix}\right|_{t=0} = \begin{pmatrix} -x_1 \\ -x_1 x_2 \end{pmatrix} = f(x).$$

Note that $f(x)$ is itself C^1 on $\mathbb{R}^2$. ∎

Not every differential equation defines a complete flow, because, as we saw in §3.5, the solutions do not necessarily exist for all time. However, if they do, then the flow is complete.

Lemma 4.2. *Let E be an open subset of $\mathbb{R}^n$, and $f : E \to \mathbb{R}^n$ a C^1 vector field such that the initial value problem $\dot{x} = f(x)$, $x(0) = x_o$, has a solution $u(t; x_o) \in E$ that exists for all $t \in \mathbb{R}$ and all $x_o \in E$. Then $\varphi_t(x_o) \equiv u(t; x_o)$ is a complete flow.*

Proof. Theorem 3.15 implies that $u(t; x_o)$ is a differentiable function of both t and x_o. Moreover, the solution is unique in any interval in which it exists. To identify the solution as a flow, the group property must be demonstrated. Choose an $s \in \mathbb{R}$ and define $x_1 = u(s, x_o)$. The initial value problem starting at x_1 has a solution that, by uniqueness, is given by the same function $u(t; x_1)$. However, uniqueness also implies that this new solution must follow the original solution; therefore,

$$u(s + t; x_o) = u(t; x_1) = u(t; u(s; x_o)).$$

This is the group property (4.4). □

4.3 Global Existence of Solutions

Theorem 3.10 (existence and uniqueness) implies that if a vector field $f : E \to \mathbb{R}^n$ is Lipschitz, then the initial value problem

$$\dot{x} = f(x), \qquad x(0) = x_o, \tag{4.6}$$

has a unique solution for t within a maximal, open interval $J = (\alpha, \beta)$ (recall Theorem 3.16). As we have noted, such a solution defines a flow, although the flow is not complete when either α or β is not infinite. Recall that a *complete* flow must obey the group property (4.4) for all t and $s \in \mathbb{R}$, and so the interval of existence must be all of $\mathbb{R}$. This makes the discussion of the global properties of the solutions of ODEs somewhat problematic.

There are several ways in which this problem can be obviated. For example, whenever the vector field f is bounded, the solutions do give a flow, as in the following theorem.

Theorem 4.3 (Bounded Global Existence). *If $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz and bounded, then the solution of* (4.6) *defines a complete flow.*

Proof. Since f is locally Lipschitz, a solution $x(t) = u(t; x_o)$ exists on some maximal, open interval (α, β). By assumption, there is an M such that $|f(x)| \leq M$. The integral equation (3.10) then gives the inequality (for $t > 0$)

$$|x(t) - x_o| \leq \int_0^t |f(x(s))|\, ds \leq Mt.$$

If β were finite, then this inequality implies that $x(t)$ is contained in the compact set $\{x : |x - x_o| \leq M\beta\}$; however, this contradicts Theorem 3.17 (unboundedness). Consequently, β is not the maximal value, and indeed there is no finite upper limit for the interval of existence. Similarly, it can be argued that α cannot be finite and therefore that the solution exists for all t. The solution defines a flow by Lemma 4.2. □

For example, the flow of the vector field $f(x) = \operatorname{sech}(x)$ on $\mathbb{R}$ is complete. Unfortunately, as shown in §3.5, the flow of an unbounded vector field such as $f(x) = x^2$ is not typically complete. Nevertheless, it is possible to show that any such flow is *equivalent* to a complete flow.

Theorem 4.4. *If $f(x)$ is locally Lipschitz on $\mathbb{R}^n$, then* (4.6) *is equivalent to*

$$\frac{dy}{d\tau} = F(y) = \frac{f(y)}{1 + |f(y)|}$$

upon reparameterizing time. The vector field F defines a flow on $\mathbb{R}^n$ since it is Lipschitz and bounded.

The use of the term "equivalence" for changing the definition of the time variable will be discussed more in §4.7.

Proof. The original equation has a solution $x(t)$ in some maximal interval (α, β). Define $y(\tau(t)) = x(t)$ using the new time variable

$$\tau = \int_0^t (1 + |f(x(s))|)\, ds. \tag{4.7}$$

Since $d\tau \big/ dt = 1 + |f(x(t))| > 0$, the transformation (4.7) is strictly monotone increasing, so it defines a one-to-one mapping τ. Moreover, the differential equation for $y(\tau)$ is

$$\frac{dy}{d\tau} = \frac{dx}{dt}\frac{dt}{d\tau} = \frac{f(x)}{1 + |f(x)|} = F(y(\tau)). \tag{4.8}$$

Using the identity $(ab - cd) = \frac{1}{2}[(a-c)(b+d) + (b-d)(a+c)]$, it is not too hard to show that the new vector field F is locally Lipschitz:

$$\begin{aligned}
|F(y) - F(x)| &= \frac{|f(x)(1+|f(y)|) - f(y)(1+|f(x)|)|}{(1+|f(x)|)(1+|f(y)|)} \\
&= \frac{1}{2}\frac{|(f(x)-f(y))(2+|f(x)|+|f(y)|)+(|f(y)|-|f(x)|)(f(x)+f(y))|}{(1+|f(x)|)(1+|f(y)|)} \\
&\leq |f(x) - f(y)| \frac{1+|f(x)|+|f(y)|)}{(1+|f(x)|)(1+|f(y)|)}.
\end{aligned}$$

Since the ratio above is bounded by one, F has the same Lipschitz constant as f. Moreover, as the new vector field F is bounded, Theorem 4.3 implies that the solutions of (4.7) exist for all time. The solution $x(t)$ must be unbounded as $t \to \alpha$ or β; consequently, the transformation τ maps J onto the infinite interval $(-\infty, \infty)$. □

Global existence also can be proved for vector fields that are globally Lipschitz.

Theorem 4.5 (Lipschitz Global Existence). *Suppose that $f(x)$ is globally Lipschitz on $\mathbb{R}^n$. Then the solutions exist for all time, and therefore define a flow.*

Proof. Beginning just as in the proof of Theorem 4.3, we obtain from the integral equation (3.10) the inequality

$$|x(t) - x_o| \leq \int_0^t |f(x(s))| ds \leq \int_0^t (|f(x(s)) - f(x_o)| + |f(x_o)|)\, ds$$

for any $0 \leq t \leq \beta$. The first term in the integral can be bounded using the global Lipschitz constant, K, for f. Suppose that β is finite; then for all $0 \leq t \leq \beta$,

$$|x(t) - x_o| \leq \beta\, |f(x_o)| + K \int_0^t |x(s) - x_o|\, ds,$$

which by the Grönwall inequality (3.30) implies that $|x(t) - x_o| \leq \beta\, |f(x_o)|\, e^{Kt}$. Hence, when $0 \leq t \leq \beta$, $x(t)$ is contained in the compact set $\left\{x : |x - x_o| \leq \beta\, |f(x_o)|\, e^{K\beta}\right\}$. However, by Theorem 3.17 (unboundedness) this is impossible, so β cannot be finite. A similar argument shows that α is not finite. □

In some cases, a system of ODEs has a singularity that gives rise to a finite interval of existence. However, we can also often use the idea of rescaling time in this case to obtain a set of equations with global solutions.

Example: Consider two point masses interacting through mutual gravitational forces, and suppose that the velocity of the particles is tangent to the line connecting their masses. Choose a reference frame fixed on one mass, and let the origin correspond to the position of this mass. Denoting the position of the second particle by $x \in \mathbb{R}$, Newton's equations for this system are then

$$\dot{x} = v, \quad \dot{v} = -\frac{K}{x^2}\operatorname{sgn}(x), \tag{4.9}$$

where $K = G(m_1 + m_2)$. This is a Hamiltonian system—recall (1.12)—on the two-dimensional phase space of position, x, and velocity, v, with energy $H = {}^1\!/_2 v^2 - K\big/|x|$. However, we must restrict our attention to the set where $x \neq 0$ to avoid a singularity in the equation; consequently, the interval of existence is finite when a collision occurs (for example, when $H < 0$). In 1920 Levi–Civita developed a transformation that *regularizes* this collision singularity (Siegel and Moser 1971). By analogy with (4.7), he defines a new time by

$$\tau = \int_0^t \frac{ds}{x(s)}.$$

To simplify the equations, Levi–Civita also defines new dynamical variables (u, w) using the transformation

$$\begin{aligned} x &= u^2 \\ v &= 2\frac{w}{u} \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} u &= \sqrt{x} \\ w &= \frac{1}{2} v\sqrt{x}, \end{aligned}$$

which is well defined for $x > 0$. Substituting these transformations into the system (4.9) gives

$$\begin{aligned} \frac{du}{d\tau} &= \frac{1}{2} v\sqrt{x} = w, \\ \frac{dw}{d\tau} &= \frac{w^2}{u} - \frac{K}{2u} = \frac{1}{2} H u, \end{aligned} \tag{4.10}$$

where $H = (2w^2 - K)\big/u^2$ is the energy in the new coordinates. Since H is a constant, this system is effectively linear and its solutions are very simple; recall (2.20). Note that this linear system is defined for all (u, w) and has a global interval of existence. When $H < 0$, the solutions to (4.10) are oscillatory, and u changes sign; the negative values of u correspond to fictitious imaginary positions of the masses.

It is much more complicated to regularize the collision of more than two point masses. The three-body collision was studied by (McGehee 1974), but the behavior near a simultaneous collision of more than three bodies is still an unresolved question. ∎

With these results, the concept of "flow" can be used to represent dynamics in most situations of interest—though with a possible reparameterization of time.

4.4 Linearization

The simplest orbit of a dynamical system is one that does not move, an

▷ *equilibrium*: A point x^* is an equilibrium of (4.6) if $f(x^*) = 0$.

Some authors use the term "critical point" or "singular point" in place of equilibrium. Neither of these is, in my opinion, good terminology, as they seem to imply that something critical or singular happens at equilibria, when in fact an equilibrium is not critical or singular at all! It is simply a place where there is no motion. Moreover, it is standard to use the term "critical point" for a point where the derivative of a function vanishes.

Example: If the ODE is a *gradient system* $\dot{x} = \nabla V(x)$, then equilibria occur at critical points of the "potential" V. Therefore, in this case the terminology "critical point" is appropriate for the equilibria. The dynamics of a gradient system can be visualized by drawing the contours of the potential, since the velocity is perpendicular to surfaces of constant V. ∎

When $f(x)$ is C^1, it is reasonable to hope that the motion in the neighborhood of an equilibrium can be studied by a Taylor series expansion of the ODE about x^*. To do this, substitute $x(t) = x^* + \delta x(t)$ into the ODE (4.6) using $f(x^*) = 0$ to obtain

$$\begin{aligned} \frac{d}{dt}\left(x^* + \delta x\right) = \frac{d}{dt}\delta x = f(x^* + \delta x) = f(x^*) + Df(x^*)\delta x + o(\delta x), \\ \frac{d}{dt}\delta x = Df(x^*)\delta x + o(\delta x), \end{aligned} \tag{4.11}$$

by Taylor's theorem. Here the notation (pronounced "little *oh* of δx") means

▷ $g(x) = o\left(f(x)\right)$ as $x \to a$ if for all $\varepsilon > 0$ there is a neighborhood $N(\varepsilon)$ of a such that $|g(x)| < \varepsilon\, |f(x)|$ for all $x \in N(\varepsilon)$.

Recall from §3.1 that a neighborhood of a point a is any set that contains an open set containing a. A similar notation is the "big oh" symbol, which means

▷ $g(x) = O\left(f(x)\right)$ as $x \to a$ if there is a neighborhood N of a and a $C \geq 0$ such that $|g(x)| < C\, |f(x)|$ for all $x \in N$.

When $f \in C^2$, then Taylor's theorem implies that the remainder term in (4.11) is actually $O(\delta x^2)$.

If we simply discard the $o(\delta x)$ terms in (4.11), we obtain an ODE called the

▷ *linearization*: If $f \in C^1(E)$, then the linearization of $\dot{x} = f(x)$ at the equilibrium $x^* \in E$ is the differential equation

$$\dot{y} = Df(x^*)y. \tag{4.12}$$

No justification, other than the desire for simplicity, has been given for neglecting the higher-order terms in (4.11); nevertheless, (4.12) does give a faithful local representation for the motion in some cases. Note that $Df(x^*) = A$ is a constant matrix and so all our techniques from Chapter 2 for solving linear systems apply. In particular, the general solution is $\Phi(t, 0)y_o$ where $\Phi(t, 0)$ is the fundamental matrix (2.46).

In §2.7 the solutions of linear ODEs were classified by their generalized eigenspaces according to the sign of the real part of the eigenvalues, resulting in the decomposition $E = E^u \oplus E^s \oplus E^c$ into the direct sum of unstable, stable, and center eigenspaces. We can now use this decomposition to classify the behavior "near" an equilibrium. We first generalize the notion of hyperbolic linear systems in §2.7 to general equilibria:

▷ *hyperbolic*: an equilibrium x^* of a C^1 vector field f is hyperbolic if none of the eigenvalues of $Df(x^*)$ have zero real part, or equivalently when E^c is empty.

Hyperbolic equilibria fall into three classes:

> ▷ *sink*: an equilibrium is a sink if all of the eigenvalues of $Df(x^*)$ have negative real parts (are in the left half of the complex plane), or equivalently when $E = E^s$;
>
> ▷ *source*: an equilibrium is a source if all of the eigenvalues of $Df(x^*)$ have positive real parts, or equivalently when $E = E^u$; and
>
> ▷ *saddle*: an equilibrium is a saddle if it is hyperbolic, but not a sink or a source, equivalently when $E = E^s \oplus E^u$.

Recall that in §2.2 an equilibrium was called a stable node when its eigenvalues are real and negative and an unstable node when they are real and positive. The classification into sink and source above includes these cases but also allows the eigenvalues to be complex. When some or all of the eigenvalues of $Df(x^*)$ are complex, we can indicate this by adding some additional modifiers to the classification:

> ▷ *focus*: there is a subspace with complex eigenvalues with nonzero real part, or
>
> ▷ *center*: there is a subspace with purely imaginary eigenvalues.

For example, a four-dimensional saddle with two pairs of eigenvalues $\lambda_{1,2} = 1 \pm 2i$ and $\lambda_{3,4} = -2 \pm 4i$ is called a saddle-focus. There are many varieties of foci, depending upon the number of complex eigenvalues. If we wish to be more precise in the classification, we can specify the dimension of each of the invariant subspaces.

Example: Consider the set of ODEs on $\mathbb{R}^3$:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = f(x, y, z) = \begin{pmatrix} x - y \\ z + y^2 \\ x + yz \end{pmatrix}. \tag{4.13}$$

Solving the three equations $f(x, y, z) = 0$ gives three equilibria, $(0, 0, 0)$, $(1, 1, -1)$, and $(-1, -1, -1)$. The Jacobian of the vector field at a general point is

$$Df = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2y & 1 \\ 1 & z & y \end{pmatrix}.$$

The characteristic polynomial of this matrix is

$$p(\lambda) = \det(\lambda I - Df) = \lambda^3 - (3y + 1)\lambda^2 - (z - 3y - 2y^2)\lambda + 1 + z - 2y^2.$$

Perhaps the hardest part of linear stability analysis is to find the roots of $p(\lambda)$. The critical points and critical values of p can be used to determine the relevant information even without explicitly finding the eigenvalues. For example, a cubic polynomial always has one real root; however, it has three real roots only if it has two real critical points, two values c_i such that $p'(c_i) = 0$, and if the signs of p at the two critical points are opposite, so $p(c_1)p(c_2) < 0$.

The first equilibrium $(0, 0, 0)$ of (4.13) has the characteristic polynomial

$$p(\lambda) = \lambda^3 - \lambda^2 + 1.$$

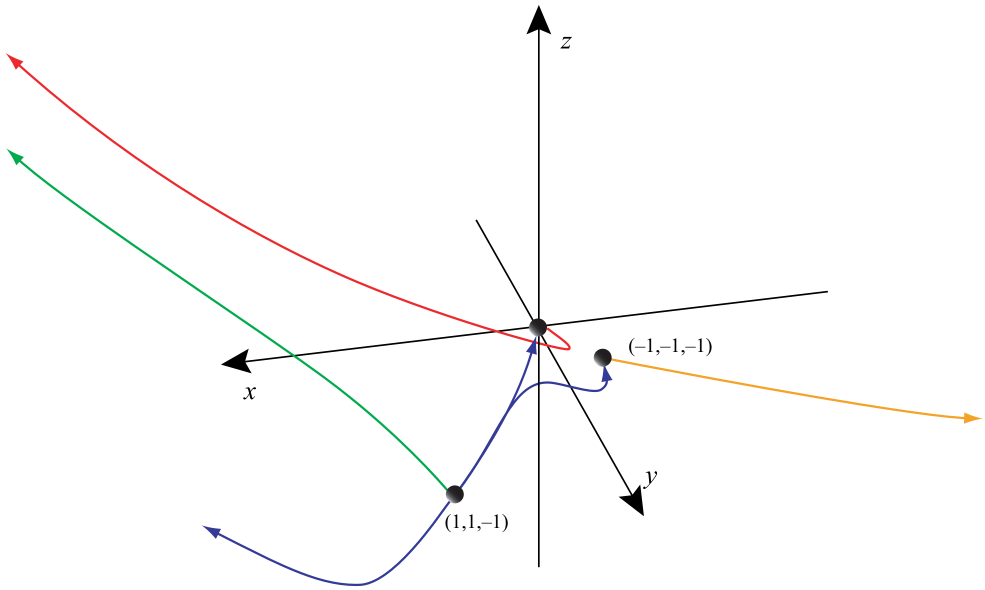

Figure 4.2. *Several orbits of the system* (4.13) *near its three equilibria*.

Since $p'(c) = 3c^2 - 2c$, there are critical points at $c_1 = 0$ and $c_2 = 2/3$, where $p(c_i) > 0$. Thus, there is only one real root. Since $p(0) = 1$, the real root, λ_1, is negative; and since $p(-1) = -1$, then $-1 < \lambda_1 < 0$. A numerical solution shows that $\lambda_1 \approx -0.7548$. The remaining roots must be complex, $\lambda_{2,3} = \alpha \pm i\beta$. The sum of the eigenvalues is $\mathrm{tr}(Df) = 1 = \lambda_1 + 2\alpha$, so that $\alpha = \frac{1}{2}(1 - \lambda_1) > \frac{1}{2}$. Numerically, $\alpha \approx 0.8774$. As a consequence, the origin is a hyperbolic saddle. Since one pair of eigenvalues is complex, it can be called a saddle-focus. Here, E^u is two-dimensional, and E^s is one-dimensional.

The second equilibrium, $(1, 1, -1)$, has the characteristic polynomial

$$p(\lambda) = \lambda^3 - 4\lambda^2 + 6\lambda - 2.$$

The critical points of p are complex, so p has only one real root. Since $p(0) < 0$ and $p(1) > 0$, then $0 < \lambda_1 < 1$. Moreover, since $Re(\lambda_{2,3}) = \alpha = \frac{1}{2}(\mathrm{tr}(Df) - \lambda_1) = \frac{1}{2}(4 - \lambda_1) > 0$, this point is a source-focus and has a three-dimensional unstable space.

Finally, the equilibrium $(-1, -1, -1)$, has characteristic polynomial

$$p(\lambda) = \lambda^3 + 2\lambda^2 - 2,$$

which has critical points at $c_1 = 0$ and $c_2 = -4/3$, where $p(c_i) < 0$, so again there is a single real root, $0 < \lambda_1 < 1$. So $\alpha = \frac{1}{2}(\mathrm{tr}(Df) - \lambda_1) = \frac{1}{2}(-2 - \lambda_1) < 0$. Thus, this point is a saddle-focus with a two-dimensional stable space and a one-dimensional unstable space. Some orbits of this system are shown in Figure 4.2. ∎

One of the major questions that we will soon address is, "To what extent does the solution of the full system *look like* the solution of the linear system?" Moreover, what is meant by *look like*? A partial answer to this will be provided by the Hartman–Grobman theorem in §4.8.

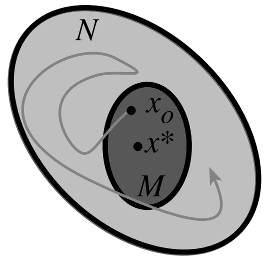

Figure 4.3. *Lyapunov stability.*

4.5 Stability

In §2.7 we said a system is *linearly stable* if it has bounded forward orbits; in other words, each orbit stays a bounded distance from the equilibrium at the origin. In that section we also defined the concepts of *spectral stability* and *asymptotic linear stability*. For nonlinear systems, these definitions are deficient: simply being bounded does not characterize the long time dynamics. A better definition of stability refers to orbits that are close: an equilibrium is stable if orbits that start "nearby" stay "nearby." Aleksandr Lyapunov (pronounced lēah·pū′·nof) (1857–1918) formalized this idea in 1892:

> ▷ *Lyapunov stability*: An equilibrium x^* of a flow φ_t is (Lyapunov) stable if for every neighborhood N of x^* there is a neighborhood $M \subset N$ such that if $x \in M$, then $\varphi_t(x) \in N$ for all $t \geq 0$.

This construction is sketched in Figure 4.3. An equilibrium that is not stable is called *unstable*.

For a metric space, Lyapunov stability is equivalent to the assertion that for every $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $x \in B_\delta(x^*)$, we have $\varphi_t(x) \in B_\varepsilon(x^*)$ for all $t \geq 0$; recall (3.1). Whenever the word "stability" is used without qualification, it should be taken to mean "Lyapunov stability."

For a one-dimensional ODE, the stability of an equilibrium, x^*, is easily investigated by examining the graph of the function f near x^*, as we discussed in §1.3. For example, if there is a $\delta > 0$ such that $f(x) < 0$ for $x \in (x^*, x^* + \delta)$ and $f(x) > 0$ for $x \in (x^* - \delta, x^*)$, then x^* is Lyapunov stable, since all points in the interval $(x^* - \delta, x^* + \delta)$ move toward x^* monotonically. This is illustrated by the middle equilibrium in Figure 4.4. Generalizing the terminology from the linear case, such a point is a *sink*. By contrast, if the signs of f are reversed, then the flow moves locally away from the equilibrium and x^* is unstable, and it is called a *source* (e.g., the leftmost equilibrium in Figure 4.4). If x^* is a zero and f has the same sign on both sides, then the point is often somewhat misleadingly called *semistable*—even though by Lyapunov's definition it is really unstable! This case corresponds to the rightmost equilibrium in Figure 4.4. If $f(x) = 0$ on an interval about x^*, then there is an interval of equilibria, and each equilibrium in the interior of this interval is stable.

These notions of sink, source, and semistable equilibria are topological: they follow without any assumptions on the smoothness of f. When $f \in C^1(\mathbb{R})$, however, these stability properties are related to hyperbolicity. For example, when $Df(x^*) \neq 0$, the equilibrium is hyperbolic; it is stable when $Df(x^*) < 0$ and unstable when $Df(x^*) > 0$.

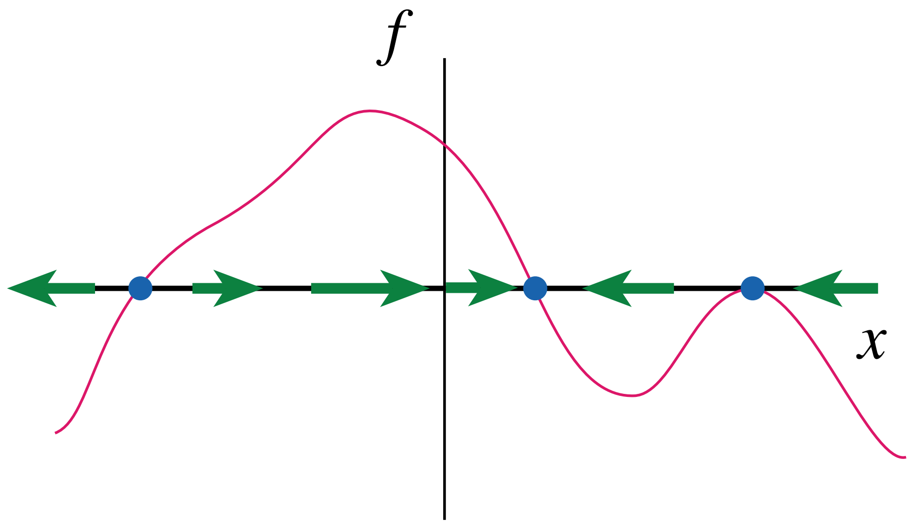

Figure 4.4. *Illustration of the three types of equilibria for a one-dimensional ODE. The left equilibrium is a source, the middle a sink, and the right is semistable.*

Example: The logistic ODE (1.7), $\dot{x} = rx(1-x)$, has an unstable equilibrium $x^* = 0$ when $r > 0$, because $Df(0) = r$, and a stable one at $x^* = 1$ where $Df(1) = -r$. Moreover, every initial condition in the interval $(0, \infty)$ moves monotonically toward 1. Indeed, for any ε, choose any $\delta \in (0, \min(\varepsilon, 1))$ and $x \in [1-\delta, 1+\delta]$; then $|\varphi_t(x) - 1| < \delta < \varepsilon$. Hence $x^* = 1$ is Lyapunov stable. ∎

Example: $f(x) = x^2 - x\cos x$. This function, shown in Figure 4.5, has precisely two zeros, $x_0 = 0$, and $x_1 = \cos(x_1) \approx 0.739085$. The solution $x(t)$ is monotone increasing if $x < x_0$ or $x > x_1$, and monotone decreasing in the interval (x_0, x_1). Accordingly, $x^* = 0$ is a stable equilibrium, while $x^* = x_1$ is unstable. ∎

A nonhyperbolic equilibrium, one for which $Df(x^*) = 0$, can be either stable or unstable. For example, the point $x = 0$ for $\dot{x} = x^2$ is semistable but not Lyapunov stable, even though all points starting with negative initial conditions asymptotically approach the origin. The problem is that there is no neighborhood containing the origin for which points stay close.

Example: Suppose $f \in C^1(\mathbb{R})$ and $Df(0) = 0$. There are four typical cases:

(a) $f(x) = -x^3$, here graphical analysis implies $x = 0$ is stable, a *sink*;

(b) $f(x) = +x^3$, unstable, a *source*;

(c) $f(x) = \pm x^2$, *semistable*; and

(d) $f(x) \equiv 0$, infinitely many equilibria. ∎

This monotonic motion toward or away from an equilibrium is specific to one-dimensional systems; higher-dimensional systems can exhibit oscillation. Moreover, even

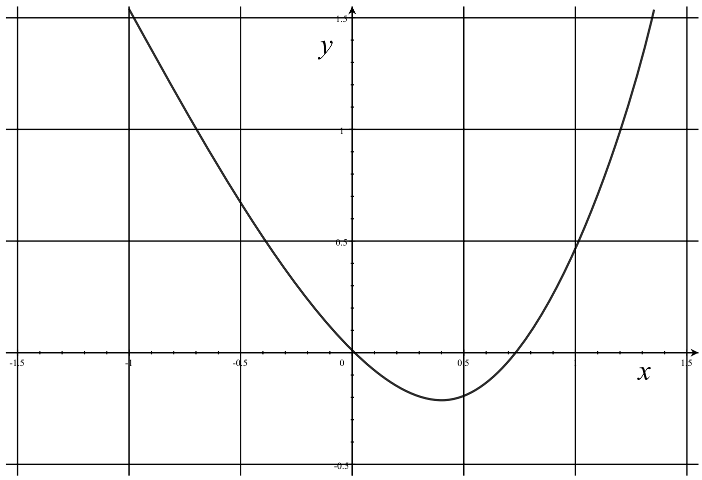

Figure 4.5. *Graph of* $f(x) = x^2 - x \cos x$.

in the linear case, the distinction between the two neighborhoods M and N is needed because the eigenvectors of a matrix are not typically orthogonal.

Example: A matrix is *normal* if it commutes with its adjoint: $[A^*, A] = 0$, where $A^* = \bar{A}^T$ is the conjugate transpose of A. It is not hard to see that the eigenspaces of a normal matrix are orthogonal. The dynamics of a stable linear system with a nonnormal matrix can exhibit a surprising temporary growth. Consider, for example,

$$\dot{x} = \begin{pmatrix} -1 & 10 \\ 0 & -2 \end{pmatrix} x \quad \Rightarrow \quad x(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} -10 \\ 1 \end{pmatrix}. \tag{4.14}$$

The general solution shows that every initial condition is attracted to the origin, so the origin should be stable. However, points that start in the disk of radius δ about the origin can leave, at least for a while. For example, setting $c_1 = 9,\ c_2 = 1$, then $x(0) = (-1, 1)$. However, the second eigenvector quickly decays, leaving a large horizontal component. Consequently, the orbit can move away from the origin for some time, as shown in Figure 4.6.

However, we can easily obtain a crude bound on $|x(t)|$, given that $|x(0)|^2 = (c_1 - 10c_2)^2 + c_2^2 \le \delta^2$. This implies that both $|c_2| \le \delta$ and $|c_1| \le 11\delta$ so that

$$\begin{aligned} |x(t)| &\le \left|c_1 e^{-t} - 10 c_2 e^{-2t}\right| + \left|c_2 e^{-2t}\right| \le |c_1|\, e^{-t} + 11\, |c_2|\, e^{-2t} \\ &\le 22\delta = \varepsilon. \end{aligned} \tag{4.15}$$

So, if we choose $\delta = \varepsilon \big/ 22$ we are guaranteed that every point that starts in the δ ball remains in the ε ball. ∎

A more stringent version of stability is the property of

▷ *asymptotic stability*: An equilibrium x^* is *asymptotically stable* if it is stable *and* there is a neighborhood N of x^* such that every point in N approaches x^* as $t \to \infty$.

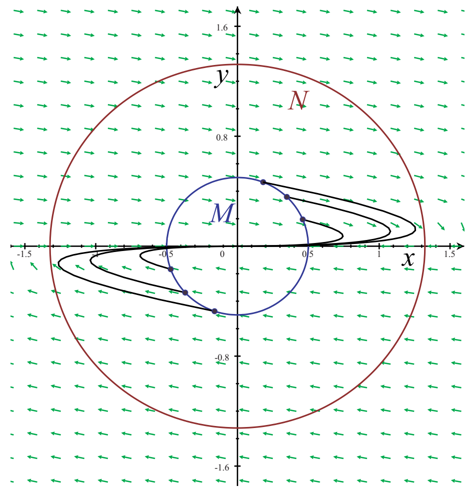

Figure 4.6. *Orbits of the system* (4.14) *that start in a neighborhood* M *never leave* N.

An asymptotically stable equilibrium is also called an *attracting equilibrium*. This is the simplest case of the concept called an *attractor*; see §4.10. Note that by this definition, an attractor must attract a neighborhood.

Example: We showed that the origin is a stable equilibrium of (4.14). Moreover, the inequality (4.15) implies that every point is asymptotic to the origin, so it is asymptotically stable as well. ∎

There are ODEs that have equilibria with a neighborhood that eventually attracts all nearby points but which is nevertheless *not* Lyapunov stable. In this case, nearby points may move a large distance from the equilibrium. A physical model ODE system is often derived to be valid only in some neighborhood of an equilibrium; consequently, when orbits move far from the equilibrium the model may no longer be valid and it would not be appropriate rely on the eventual return to define asymptotic stability.

Example: Consider the system

$$\begin{aligned} \dot{r} &= r(1-r), \\ \dot{\theta} &= \sin^2(\theta/2), \end{aligned} \tag{4.16}$$

where (r, θ) are polar coordinates in the plane. As shown in Figure 4.7, there are two equilibria, the origin and $(1, 0)$. The origin is unstable; indeed the r dynamics is decoupled from the θ dynamics, and graphical analysis immediately shows that every $r > 0$ is asymptotic to $r = 1$. Similarly the θ equation is uncoupled and since $\sin^2(\theta/2) \geq 0$, the point $\theta = 0$ is "semistable." However, since θ is a periodic coordinate, even the points with $\theta = \delta > 0$, which move away from the equilibrium point, will eventually return to $\theta = 0$. Therefore, every initial condition in $\mathbb{R}^2$ except the origin is attracted to the point $(1, 0)$. However, this

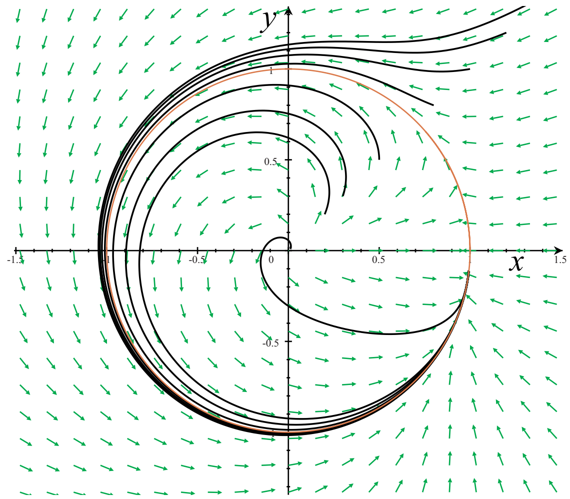

Figure 4.7. *Phase space of the example* (4.16).

point is not Lyapunov stable since for any $\varepsilon < 2$, there are nearby points—for example $(1, \delta)$—that leave the ball of radius ε about the equilibrium. ∎

Example (Vinograd): A more complicated example of this behavior was given in (Vinograd 1957):

$$\dot{x} = \frac{x^2(y-x)+y^5}{r^2(1+r^4)}, \quad \dot{y} = \frac{y^2(y-2x)}{r^2(1+r^4)}, \tag{4.17}$$

where r is the polar radius, $r^2 = x^2 + y^2$. To analyze this system, first note that the origin is the only equilibrium, since $\dot{y} = 0$ implies either $y = 0$ or $y = 2x$. In the latter case if $\dot{x} = 0$ as well, then

$$x^3 + 32x^5 = 0 \ \Rightarrow x = 0 \text{ or } x^2 = -1/32.$$

So the only real solution is $x = y = 0$. Note that $\dot{y}|_{y=0} = 0$, so the line $y = 0$ is invariant. On this line x is governed by $\dot{x}|_{y=0} = -x/(1+x^4)$; therefore, since $\mathrm{sgn}(\dot{x}) = -\mathrm{sgn}(x)$ and $\dot{x} \neq 0$ unless $x = 0$, $x(t)$ monotonically moves toward the origin, so that the origin attracts all points on this line. It is much harder to show that every point in the plane approaches the origin as $t \to \infty$, but a numerical solution (shown in Figure 4.8) indicates that this is so. More interestingly, the picture indicates that many orbits in any δ-ball leave the ball $B_{1/2}(0)$ no matter how small δ is chosen. In fact, it seems that there is a family of *homoclinic loops* from the origin, i.e., orbits that leave the origin and go a finite distance away before returning as $t \to \infty$ (see §5.2). These loops are what prevent the origin from being an attractor. The behavior of this system near the origin is studied in §6.2. ∎

When f is C^1, the local behavior near an equilibrium is often governed by the linearization, (4.12). For example, asymptotic linear stability is sufficient to imply asymptotic

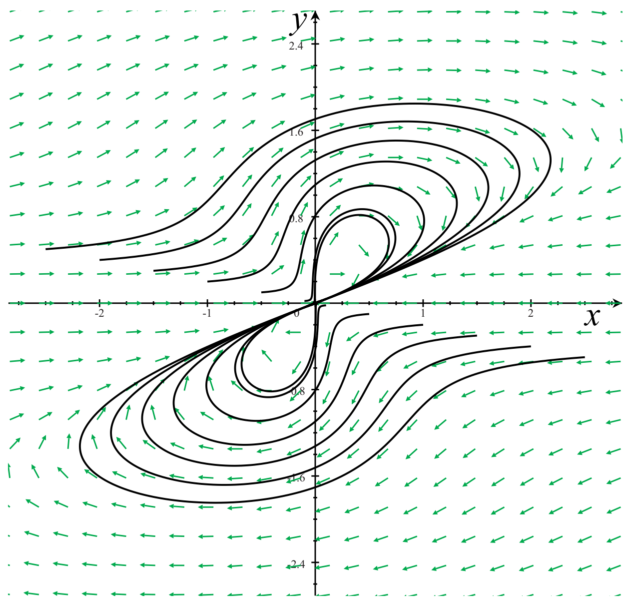

Figure 4.8. *Phase plane of the Vinograd example* (4.17).

stability of the equilibrium for the nonlinear system, if it is differentiable. The main point is that in this case we can extract the nonlinear part of f near x^* by writing

$$f(x) = Df(x^*)(x - x^*) + g(x - x^*).$$

The assumption that f is C^1 is sufficient to guarantee that the remainder term is small, i.e., that $g(\delta x) = o(\delta x)$. This follows from the definition of the derivative

$$0 = \left[\lim_{\delta x_j \to 0} \frac{f_i(x^* + \delta x_j) - f_i(x^*)}{\delta x_j} - (Df)_{ij}(x^*) \right] = \lim_{\delta x_j \to 0} \frac{g_i(\delta x_j)}{\delta x_j}.$$

Note that if $f(x)$ is C^2, then $g(\delta x) = o(\delta x^2)$, by the Taylor remainder theorem. However, we will not need this additional assumption to prove the desired result.[22]

Theorem 4.6 (Asymptotic Linear Stability implies Asymptotic Stability). *Let $f : E \to \mathbb{R}^n$ be C^1 and have an equilibrium x^* such that all the eigenvalues of $Df(x^*)$ have real parts less than zero. Then x^* is asymptotically stable.*

Proof. Rewrite the differential equations using $y = x - x^*$, defining $A = Df(x^*)$, and $g \equiv f(x) - A(x - x^*)$, to obtain

$$\dot{y} = Ay + g(y). \tag{4.18}$$

[22]This theorem also follows from either the Hartman–Grobman or the stable manifold theorem; see below.

Variation of parameters can be used to obtain an integral equation for the solution. Let $y = e^{tA}\eta(t)$, and substitute this into the ODE (4.18) to obtain $\dot{\eta} = e^{-tA}g(y(t))$. Formally integrating this equation and substituting again for y gives the integral equation:

$$y(t) = e^{tA}y_o + \int_0^t e^{(t-s)A}g(y(s))ds. \tag{4.19}$$

By assumption, there is an α such that if λ is any eigenvalue of A, then $\mathrm{Re}(\lambda) < -\alpha < 0$. The estimate (2.44) in §2.7 implies that for any vector v there is a $K \geq 1$ such that

$$\left|e^{tA}v\right| \leq Ke^{-\alpha t}|v|, \quad t \geq 0. \tag{4.20}$$

Since f is C^1, then $g(y) = o(y)$, so, for any ε there is a δ such that if $y \leq \delta$, $|g(y)| \leq \varepsilon|y|$, and thus from (4.19) using (4.20) we obtain

$$|y(t)| \leq Ke^{-\alpha t}|y_o| + K\varepsilon \int_0^t e^{-\alpha(t-s)}|y(s)|\,ds.$$

Let $\xi(t) = e^{\alpha t}|y(t)|$, and use Grönwall's Lemma 3.13 to obtain

$$\xi(t) \leq K\delta + K\varepsilon\int_0^t \xi(s)ds \quad \Rightarrow \quad \xi(t) \leq K\delta e^{K\varepsilon t} \quad \Rightarrow \quad |y(t)| \leq K\delta e^{-(\alpha - K\varepsilon)t}.$$

Hence, providing $\varepsilon < \alpha/K$, then $|y| \to 0$ and stays bounded below $K\delta$ for all $t \geq 0$. In conclusion, if M is the ball of radius δ, then N is the ball of radius $K\delta$. □

Example: The origin is an equilibrium of the system

$$\begin{aligned}\dot{x} &= -x - y - r^2,\\ \dot{y} &= x - y + r^2,\end{aligned}$$

where r is the polar radius. The origin is a stable focus since

$$Df(0,0) = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$$

has eigenvalues $\lambda = -1 \pm i$. To show that adding nonlinear terms does not change the topological character, we want to construct an attracting neighborhood of the origin. To study this system, it is easier to use the differential equation for r.[23] Noting that $2r\dot{r} = 2x\dot{x} + 2y\dot{y}$

$$\dot{r} = \frac{1}{r}\left(x\left(-x - y - r^2\right) + y(x - y + r^2)\right) = r(-1 + y - x).$$

Since $-r \leq x, y \leq r$, then $y - x \leq 2r$. If $r < 0.5$, then $-1 + y - x < 0$, and so $\dot{r} < 0$ at any point in the open disk of radius $1/2$. This implies that the origin is asymptotically stable, because r is monotonically decreasing. Note that there is another equilibrium point at $(-1, 0)$. This equilibrium has eigenvalues $\lambda = \pm\sqrt{2}$ and is therefore a saddle. Orbits near the saddle can go to infinity. ■

[23]We will find this technique extremely useful in our study of the global structure of flows in the plane in Chapter 6.

4.6 Lyapunov Functions

Lyapunov devised another technique that can potentially show that an equilibrium is stable—the construction of what is now called a "Lyapunov function." An advantage of this method is that it can sometimes prove stability of a nonhyperbolic equilibrium; a disadvantage is that there is no straightforward construction of Lyapunov functions.

Lyapunov functions are nonnegative functions that decrease in time along the orbits of a dynamical system:

> $\triangleright$ *Lyapunov function*: A continuous function $L : \mathbb{R}^n \to \mathbb{R}$ is a (*strong*) Lyapunov function for an equilibrium x^* of a flow φ_t on $\mathbb{R}^n$ if there is an open neighborhood U of x^* such that $L(x^*) = 0$, $L > 0$ for $x \neq x^*$, and
>
> $$L(\varphi_t(x)) < L(x) \quad \forall\, x \in U \setminus \left\{x^*\right\} \text{ and } t > 0. \tag{4.21}$$

The function L is a *weak* Lyapunov function if (4.21) is replaced by $L\,(\varphi_t(x)) \leq L(x)$.

Typically, L is a C^1 function and (4.21) can be guaranteed by requiring that $dL\big/dt\ < 0$. This can be computed using the chain rule:

$$\frac{dL}{dt} = \nabla L(x) \cdot f(x). \tag{4.22}$$

Consequently, in the smooth case, the condition that L is a Lyapunov function is that its gradient vector points in a direction opposed to that of the vector field f.

If such a nonincreasing function can be found, the equilibrium is stable.

Theorem 4.7 (Lyapunov Functions). *Let x^* be an equilibrium point of a flow $\varphi_t(x)$. If L is a weak Lyapunov function in some neighborhood U of x^*, then x^* is stable. If L is a strong Lyapunov function, then x^* is asymptotically stable.*

Proof. First we prove stability. We can assume that $x^* = 0$ without loss of generality. Choose any ε small enough so that $B_\varepsilon(0) \subset U$ and define $m = \min\{L(x) : |x| = \varepsilon\}$, as in Figure 4.9. The constant m exists because $B_\varepsilon(0)$ is compact and, since L is positive definite, $m > 0$. Since L decreases as $x \to 0$,there exists a $\delta < \varepsilon$ such that $L(x) < m$ for all $x \in B_\delta(0)$. Since L is nonincreasing along orbits then $L(\varphi_t(x)) < m$ for all $x \in B_\delta(0)$. Therefore, since L remains less than the minimum on $|x| = \varepsilon$, $\varphi_t(x) \in B_\varepsilon(0)$. Consequently, the origin is stable.

Now we prove asymptotic stability. If $x \in B_\delta(0)$, then $\varphi_t(x) \in B_\varepsilon(0)$ for all positive time. Since $B_\varepsilon(0)$ is compact, the Bolzano–Weierstrass theorem, Theorem 3.1, implies that for any sequence $t_i \to \infty$, the sequence $\varphi_{t_i}(x)$ must have limit points. Suppose one of these limit points is not the origin, i.e., there is a sequence of times $t_n \to \infty$ such that $\varphi_{t_n}(x) \to z \neq 0$. By continuity $L(\varphi_{t_n}(x)) \to L(z)$, and since L is strictly decreasing, the sequence of values must decrease monotonically with n:

$$L(\varphi_{t_n}(x)) > L(\varphi_{t_{n+1}}(x)) > \cdots > L(z). \tag{4.23}$$

Now consider the orbit, $\varphi_s(z)$ of the limit point z. Again, since z is not the equilibrium, $L(\varphi_t(z)) < L(z)$ for any positive s, and hence by continuity $L(\varphi_{t_n+s}(x)) \to L(\varphi_s(z)) < L(z)$. This implies for large enough n that, since $x(t_n)$ is arbitrarily close to z, $L(\varphi_{t_n+s}(x)) <$

Proof. By assumption every orbit in M is bounded; this implies the function λ defined by

$$\lambda(x) = \sup_{t\geq 0} \left|\varphi_t(x) - x^*\right|$$

for $x \in M$ is continuous. Indeed, asymptotic stability implies that for any ρ there is a time $T(\rho)$ such that $|\varphi_t(x) - x^*| < \rho$ whenever $t > T(\rho)$. As a consequence, the supremum in the definition need only be taken over a finite interval of time. Moreover, since $\varphi_t(x)$ is continuous for any fixed time, the norm $|\varphi_t(x) - x^*|$ is also continuous. To show that λ is also continuous, take any $x, y \in M \setminus B_\rho(x^*)$; then

$$\begin{aligned}
|\lambda(x) - \lambda(y)| &= \left| \sup_{0\leq t\leq T(\rho)} |\varphi_t(x) - x^*| - \sup_{0\leq t\leq T(\rho)} |\varphi_t(y) - x^*| \right| \\
&\leq \left| \sup_{0\leq t\leq T(\rho)} (|\varphi_t(x) - x^*| - |\varphi_t(y) - x^*|) \right| \\
&\leq \left| \sup_{0\leq t\leq T(\rho)} (|\varphi_t(x) - \varphi_t(y)|) \right|.
\end{aligned}$$

Since $\varphi_t(x)$ is continuous as a function of x, for any $\varepsilon > 0$ there is a $\delta(t) > 0$ such that if $|x - y| < \delta(t)$, then $|\varphi_t(x) - \varphi_t(y)| < \varepsilon$. Therefore, $|\lambda(x) - \lambda(y)| < \varepsilon$ for the choice $\delta = \inf_{0\leq t\leq T(\rho)} \delta(t)$, and $|x - y| < \delta$, which implies continuity.

Notice also that $\lambda(x^*) = 0$, and otherwise that $\lambda(x) > 0$, so it satisfies two of the properties that are needed to be a strong Lyapunov function. Moreover, $\lambda(\varphi_t(x)) \leq \lambda(x)$ when $t \geq 0$, because

$$\begin{aligned}
\lambda(\varphi_t(x)) &= \sup_{s>0} |\varphi_s(\varphi_t(x)) - x^*| = \sup_{s>0} |\varphi_{s+t}(x) - x^*| \\
&= \sup_{s>t} |\varphi_s(x) - x^*|,
\end{aligned}$$

and the last expression is definitely not larger than $\lambda(x)$. Consequently, λ is a weak Lyapunov function. We now show that (4.25) is a strong Lyapunov function. Note that for any $t > 0$,

$$L(\varphi_t(x)) = \int_0^\infty e^{-s}\lambda(\varphi_{s+t}(x))ds \leq \int_0^\infty e^{-s}\lambda(\varphi_s(x))ds = L(x).$$

If the two sides of this inequality were equal, then $\lambda(\varphi_{t+s}(x)) = \lambda(\varphi_s(x))$ for all $s > 0$. However, this is impossible since if we set $t = (n-1)s$, then we would have $\lambda(\varphi_{ns}(x)) = \lambda(\varphi_s(x))$ for all n. This cannot happen since if $x \neq x^*$, $\lambda(\varphi_s(x)) \neq 0$, but $\varphi_{ns}(x) \to x^*$, so that $\lambda(\varphi_{ns}(x)) \to 0$. □

Although this theorem guarantees that a strong Lyapunov function exists for an asymptotically stable equilibrium, it is not possible to construct it in general unless the flow can be obtained analytically—in which case there is no reason to find L! However, there are cases in which it is not hard to find a Lyapunov function and for which stability is not obvious (see Exercise 8).

Example: The Lorenz system, (1.33), is

$$\begin{aligned}\dot{x} &= \sigma(y-x),\\ \dot{y} &= rx - y - xz,\\ \dot{z} &= xy - bz,\end{aligned} \tag{4.26}$$

where we assume, as in the physical model, that the parameters r, σ, and b are positive. The equilibrium at the origin has linear stability determined by the Jacobian

$$Df(0) = \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}.$$

The z direction corresponds to an eigenvector with eigenvalue $\lambda = -b$ and is therefore always attracting for $b > 0$. The other two eigenvalues are determined by

$$\lambda^2 + (\sigma + 1)\lambda + \sigma(1 - r) = 0.$$

This implies that the $x - y$ plane is attracting when $r < 1$ but becomes a saddle for $r > 1$. Consequently, when $r < 1$ the origin is asymptotically stable and when $r > 1$ it is unstable. Linear analysis cannot tell us what happens when $r = 1$.

We now attempt to construct a Lyapunov function. Beginning with a general quadratic in (x, y, z), one can fairly quickly see that the function

$$L = \frac{1}{2}\left(\frac{x^2}{\sigma} + y^2 + z^2\right)$$

will work. Differentiation yields

$$\begin{aligned}\frac{dL}{dt} &= \left(yx - x^2\right) + ryx - y^2 - xyz + zxy - bz^2\\ &= (r+1)xy - \left(x^2 + y^2 + b^2z^2\right)\\ &= -\left(x - \frac{r+1}{2}y\right)^2 - \left(1 - \frac{(r+1)^2}{4}\right)y^2 - bz^2,\end{aligned}$$

where we completed the square on the first two terms to get the third line. Therefore, when $r < 1$ and $b > 0$, this is negative definite, confirming again that the origin is asymptotically stable. Interestingly, this analysis applies for *any* values of (x, y, z), so that the origin is globally asymptotically stable.

When $r = 1$, $dL/dt = 0$ on the line $Z = \{(x, y, z) : x = y, z = 0\}$. This means that L is not a strong Lyapunov function. However, the following argument will imply that since this set is not invariant (because $dz/dt\big|_Z \neq 0$), the origin is asymptotically stable in this case as well! ∎

As in the previous example, it is sometimes possible to conclude that the equilibrium is asymptotically stable for the case that L is a weak Lyapunov function, provided that we know something about the dynamics on the set where $dL/dt = 0$.

Theorem 4.9 (LaSalle's Invariance Principle). *Suppose x^* is an equilibrium for $\dot{x} = f(x)$ and suppose that L is a weak Lyapunov function on some compact, forward-invariant* *neighborhood U of x^*. Let $Z = \{x \in U : dL/dt = 0\}$ be the set where L is not decreasing. Then if x^* is the largest forward invariant subset of Z, it is asymptotically stable and attracts every point in U.*

Proof. For any $x \in U$, suppose z is a limit point of the trajectory $x(t) \in U$. Then $L(\varphi_s(z)) = L(z)$ for all $s > 0$, since if $L(\varphi_s(z)) < L(z)$ we would have a contradiction with the inequalities in (4.23). Consequently, $\varphi_s(z) \in Z$ for all $s > 0$, so that z must be forward invariant, and therefore, by assumption, $z = x^*$. □

Example: A slightly more realistic model than the logistic equation (1.7) adds "delay," modeling the fact that the gestation period is nonzero, and so the competition that affects current births is in the past. One type of delay is to introduce a second variable y that represents the population at an earlier era. The model then becomes

$$\begin{aligned}\dot{x} &= rx\,(1-y)\,,\\ \dot{y} &= b(x-y).\end{aligned}$$

Note that at equilibrium $y = x$ and so $x = 0$ or $x = 1$ as for (1.7). Our goal is to show that the point $(1, 1)$ is the limit of all initial conditions in the positive quadrant. First note that the positive quadrant is forward invariant. To leave it, the orbit would have to pass through the x- or y-axis. When $x = 0$, $\dot{x} = 0$, so this is an invariant line. When $y = 0$, then $\dot{y} \geq 0$, so the orbit cannot cross to negative y.

We next transform to coordinates centered at the equilibrium of interest. Let $(\xi, \eta) = (x-1, y-1)$ so that

$$\begin{aligned}\dot{\xi} &= -r\eta\,(1+\xi)\,,\\ \dot{\eta} &= b(\xi-\eta).\end{aligned}$$

Note that $(0, 0)$ is a linearly stable equilibrium for this equation when b and r are positive since then $\mathrm{tr}(Df(0,0)) = -b < 0$ and $\det(Df(0,0)) = rb > 0$ (recall §2.2). A simple quadratic function will not work as a Lyapunov function for this system, nor will any polynomial of finite order. However, after some guesswork—see (MacDonald 1978)—a Lyapunov function can be found:

$$L(\xi,\eta) = \xi - \ln(1+\xi) + \frac{r}{2b}\eta^2.$$

Note that $L(0, 0) = 0$, and that since $\xi - \ln(1+\xi) \geq 0$ when $\xi > -1$, then L is positive. Furthermore, differentiation gives

$$\frac{dL}{dt} = -r\eta(\xi+1)\left(1 - \frac{1}{\xi+1}\right) + r\eta(\xi-\eta) = -r\eta^2.$$

Accordingly, L is strictly decreasing except on the set $Z = \{(\xi,\eta), \eta = 0, \xi > -1\}$. However, the equations of motion imply that the only invariant point in Z is the origin since $\dot{\eta}|_Z = b\xi \neq 0$ otherwise. Therefore, according to LaSalle's invariance principle, $(0, 0)$ attracts the orbits of all initial conditions with $\xi > -1$. Equivalently, in the original coordinates, the point $(1, 1)$ is the forward limit of all points in the right half-plane. ∎

Hamiltonian systems—recall §1.4—often have Lyapunov functions. Suppose that $H : \mathbb{R}^2 \to \mathbb{R}$, and consider the Hamiltonian system

$$\dot{x} = \frac{\partial H}{\partial y}, \quad \dot{y} = -\frac{\partial H}{\partial x}. \tag{4.27}$$

The value of $H(x, y)$ typically represents the "energy" of the system. It is constant along trajectories, because

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} = \frac{\partial H}{\partial x}\frac{\partial H}{\partial y} - \frac{\partial H}{\partial y}\frac{\partial H}{\partial x} \equiv 0. \tag{4.28}$$

Therefore, if $H(x_o, y_o) = E$, then so does $H(x(t), y(t))$. If (x^*, y^*) is an equilibrium, then the function

$$L(x, y) = H(x, y) - H(x^*, y^*)$$

is zero at the equilibrium and constant along trajectories; consequently, if it can be shown that L is positive in some neighborhood of the equilibrium, then it is a weak Lyapunov function.

Example: Consider the system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x - 3ax^2. \end{aligned} \tag{4.29}$$

These equations have the form (4.27), since if $y = \partial H \big/ \partial y$, then $H(x, y) = {}^1\!/_2 y^2 + V(x)$, for an arbitrary function V. Similarly, demanding that $x - 3ax^2 = -\partial H \big/ \partial x$ gives $H(x, y) = T(y) - {}^1\!/_2 x^2 + ax^3$, for an arbitrary function T. These two equations are consistent, implying that (4.29) is Hamiltonian and we obtain $H(x, y) = {}^1\!/_2(y^2 - x^2) + ax^3$.

The system (4.29) has two equilibria, $(0, 0)$ and $(1\big/3a, 0)$. The first is a saddle, and the second is a center. The Hamiltonian provides a Lyapunov function in a neighborhood of the center. We can see this most easily by shifting coordinates, defining $\xi = x - 1\big/3a$ to obtain

$$H = 1/2\left(y^2 + \xi^2\right) + a\xi^3 + H\left(1\big/3a, 0\right).$$

Therefore, for ξ small enough, H has contours about $y = \xi = 0$ that are approximately circular. In conclusion, $L = {}^1\!/_2\left(y^2 + \xi^2\right) + a\xi^3$ is a weak Lyapunov function, and the equilibrium $\left(1\big/3a, 0\right)$ is a "topological center"—see §6.2. ∎

We will discuss more examples of this type in §5.1 (see also Exercise 8).

Although Hamiltonian systems correspond to "conservative" dynamics, engineering systems often have damping.

Example: Suppose $x \in \mathbb{R}^n$ are coordinates and $y \in \mathbb{R}^n$ are the conjugate momenta, with the Hamiltonian $H(x, y) = {}^1\!/_2|y|^2 + V(x)$. Here, $V(x)$, the potential energy, gives rise to the force $F = -\nabla V$. This system is conservative; the simplest model for damping is an additional force proportional to the momentum, which gives the set of equations

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\nabla V(x) - \gamma y, \end{aligned} \tag{4.30}$$

where γ is the damping coefficient. The "energy" of this system is given by the function $H(x, y)$. If we assume that $\nabla V(0) = 0$, so that the origin is an equilibrium, then the origin

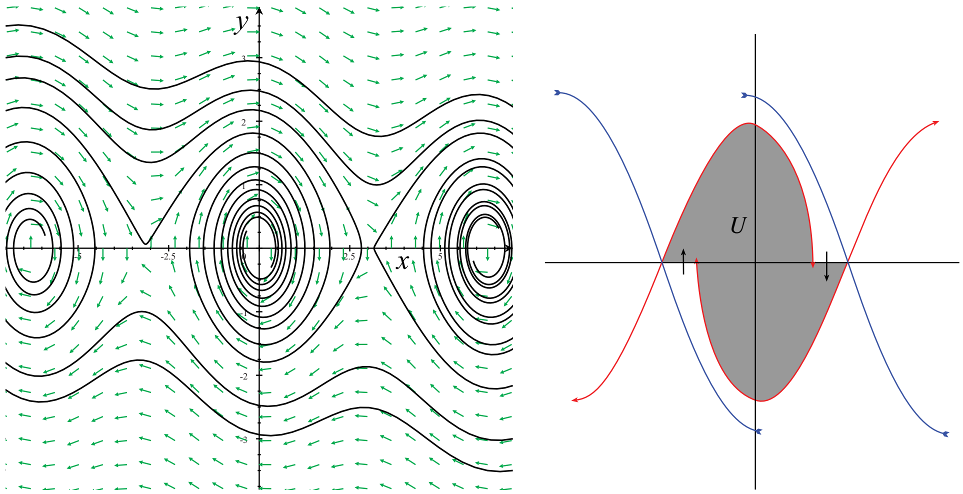

Figure 4.10. *Phase space of the damped pendulum* (4.30) *with* $V(x) = -\cos x$*, and* $\gamma = 0.1$*.* V *has critical points on the* x*-axis at* $n\pi$*. The points* $(2k\pi, 0)$ *are asymptotically stable, while* $((2k+1)\pi, 0)$ *are saddles. On the right is shown a forward invariant region* U *enclosing the origin.* U *is bounded by pieces of the unstable manifolds (see* §5.1*) of the saddles at* $x = \pm\pi$ *and by part of the* x*-axis. To prove that* U *exists, we would have to show that the unstable manifolds (see Chapter* 5*) of the saddles first cross the* x*-axis in the interval* $(-\pi, \pi)$*.*

is a critical point of H, since

$$\nabla H = (\nabla V(x), y)^T.$$

Moreover, when $D^2V(0)$ is a positive definite matrix, the Hessian matrix,

$$D^2H(0) = \begin{pmatrix} D^2V(0) & 0 \\ 0 & I \end{pmatrix},$$

is also positive definite so that the origin is a minimum of H. In this case, the contours of H are closed near the origin. Moreover,

$$\frac{dH}{dt} = y \cdot (-\nabla V - \gamma y) + y \cdot \nabla V = -\gamma |y|^2 \le 0;$$

therefore, the origin is stable.

If 0 is the only critical point of V, then LaSalle's invariance principle implies that the origin is asymptotically stable. The set for which $dH/dt = 0$ is $Z = \{(x, y) : y = 0\}$. Now since $\dot{y}|_Z = -\nabla V(x)$, whenever x is not a critical point of V, then $\dot{y} \neq 0$ on Z. We can conclude that if 0 is the only critical point of V, the only invariant subset of Z is the origin.

The analysis above could be generalized to the case where there are more critical points of V if it could be proved that there exists a neighborhood, U, of the origin—like that depicted in Figure 4.10—that does not include other critical points and that is forward invariant. ∎

4.7 Topological Conjugacy and Equivalence

An important task in dynamical systems is to determine whether two dynamical systems that seemingly look "different" are actually the same but are just written in different forms. A system that looks complicated may actually be quite simple in a different coordinate system. A classification of equivalent systems will considerably reduce the work to be done, for example, in bifurcation theory (see Chapter 8). Moreover, the study of these equivalence classes leads to notions of sensitivity of dynamics to modification of the system—what is called structural stability.

There are several different notions of equivalence, depending upon the degree of smoothness required for the transformation. The definitions require some notions from basic set theory and topology. Suppose that A and B are two topological spaces (recall §3.1). A map h: A → B is

▷ *surjective* or *onto* if for every $b \in B$, there is at least one $a \in A$ such that $h(a) = b$,

▷ *injective* or *one-to-one* if whenever $h(a) = h(a')$, then $a = a'$, and

▷ *bijective* if it is both surjective and injective.

Note that a bijective map has an inverse: since for each b there is exactly one a such that $b = h(a)$, the map $h^{-1} : B \to A$ is defined by setting $a = h^{-1}(b)$. Note that h^{-1} is both a left and a right inverse for h: $h(h^{-1}(b)) = b$ and $h^{-1}(h(a)) = a$. These notions are used to define one of the most fundamental concepts in topology:

▷ *homeomorphism*: A map $h : A \to B$ is a *homeomorphism* if it is continuous, is bijective, and has a continuous inverse.

For example, the map $h : (0, \infty) \to (0, 1)$ defined by $h(x) = 1\big/(1 + x^2)$ is a homeomorphism. Similarly, the map $f : \mathbb{S} \to \mathbb{S}$ defined by

$$f(\theta) = \theta + a\cos\theta \tag{4.31}$$

is a homeomorphism only when $|a| < 1$, since it is otherwise not one-to-one; see Figure 4.11.[24]

Topology declares that two spaces are equivalent if there is a homeomorphism from one to the other. It is this notion that implies that a mug of coffee and a doughnut are the "same" (though one gives you a buzz from caffeine and the other from sugar). Conversely, if it can be shown that there is no homeomorphism from one space to another, then they are topologically distinct spaces.

It is natural to also define a notion of "smooth" equivalence:

▷ *diffeomorphism*: A map $f : A \to B$ is a diffeomorphism if it is a C^1 bijective map with a C^1 inverse.

[24]Challenge for the topologically inclined: find an example of a continuous, bijective map that is not a homeomorphism. At least one of the spaces must have an exotic topology, because every continuous, bijective map from a compact space to a Hausdorff space is a homeomorphism (Hocking and Young 1961).

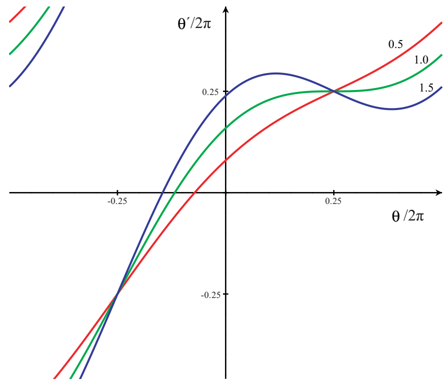

Figure 4.11. *The function* (4.31) *for* $a = 0.5$, 1.0, *and* 1.5. *The last case is not a homeomorphism since the graph is not monotone*.

For example, $f : \mathbb{R} \to \mathbb{R}$, given by $f(x) = x + \frac{1}{2}\sin x$ is a diffeomorphism, but $f(x) = x^3$ is not because its inverse, $f^{-1}(x) = x^{\frac{1}{3}}$, is not C^1. Note that every diffeomorphism is also a homeomorphism. Recall from §4.2 that a flow is a C^1 bijection from the phase space to itself, and thus the map φ_t for each time t is a diffeomorphism.

With these definitions in our toolbox, we are now prepared to understand the key notion of equivalence of two flows,

▷ *topological conjugacy*: Two flows $\varphi_t : A \to A$ and $\psi_t : B \to B$ are conjugate if there exists a homeomorphism $h : A \to B$ such that for each $x \in A$ and $t \in \mathbb{R}$

$$h(\varphi_t(x)) = \psi_t(h(x)). \tag{4.32}$$

It is clear that for such a homeomorphism to exist, A and B must be topologically equivalent spaces. Often, two systems are simply said to be *conjugate* as a shorthand for topologically conjugate. A diagram that represents (4.32) is

$$\begin{array}{ccc} x & \xrightarrow{\varphi_t} & \varphi_t(x) \\ h\downarrow & & \downarrow h \\ y & \xrightarrow{\psi_t} & \psi_t(y) \end{array} .$$

The two paths in this diagram, $x \xrightarrow{h} y \xrightarrow{\psi_t} \psi_t(y)$ and $x \xrightarrow{\varphi_t} \varphi_t(x) \xrightarrow{h} \psi_t(y)$, which represent the right- and left-hand sides of (4.32), respectively, must give the same result, namely, $\psi_t(h(x))$. We say, in this case, that the “diagram commutes.”

Example: The flow on $\mathbb{R}$ generated by $\dot{x} = -x$ is $\varphi_t(x) = xe^{-t}$. Under the homeomorphism $y = h(x) = x^3$, this is equivalent to the new flow

$$\psi_t(y) = (xe^{-t})^3 = ye^{-3t}.$$

This is the solution of the linear equation $\dot{y} = -3y$. Consequently, these two ODEs are topologically conjugate. ∎

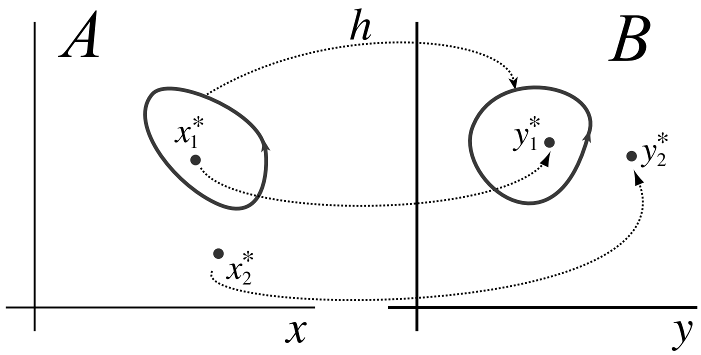

Figure 4.12. *Orbits of conjugate systems must be in a one-to-one correspondence.*

Conjugacy implies that each trajectory of ψ corresponds to a trajectory of φ, and vice versa. For example, if x^* is an equilibrium of φ, then since $\varphi_t(x^*) = x^*$ for all t, $\psi_t(h(x^*)) = h(x^*) = y^*$ and so y^* is an equilibrium of ψ. Thus, h provides a one-to-one correspondence between the equilibria of two conjugate flows. Similarly, if $\varphi_t(x_o)$ is a periodic orbit of φ with period T, i.e., $\varphi_{t+T}(x_o) = \varphi_t(x_o)$, then $\psi_t(y_o) = h(\varphi_t(x_o)) = h(\varphi_{t+T}(x_o)) = \psi_{t+T}(y_o)$, so $\psi_t(y_o)$ is also a periodic orbit of ψ with the same period; see Figure 4.12.

Topological conjugacy can be too restrictive a condition because, in addition to the fact that trajectories "look" the same in phase space, (4.32) implies that the curves have identical temporal parameterizations. A slightly more general notion that still captures the shape and direction of the flows as curves in phase space is

> ▷ *topological equivalence*: Two flows $\varphi_t : A \to A$ and $\psi_t : B \to B$ are equivalent if there exists a homeomorphism $h : A \to B$ that maps the orbits of φ onto the orbits of ψ and preserves the direction of time. That is, there is a map $\tau : A \times \mathbb{R} \to \mathbb{R}$ that is monotone increasing with t and
>
> $$h(\varphi_{\tau(x,t)}(x)) = \psi_t(h(x)). \tag{4.33}$$

Example: If we temporarily relax the requirement that a flow exist for all time, then

$$\psi_t(y) = \frac{y}{1+ty}$$

is the flow corresponding to the ODE $\dot{y} = -y^2$. For $y \in \mathbb{R}^+$, it exists only on the interval $t \in (y^{-1}, \infty)$. This flow is equivalent to $\varphi_t(x) = xe^{-t}$ under the transformations $h(x) = x$, and $\tau(x,t) = \ln(1+xt)$, since

$$h(\varphi_{\tau(x,t)}(x)) = xe^{-\ln(1+xt)} = \frac{x}{1+xt} = \psi_t(h(x)).$$

Note that the orbits of ψ are qualitatively the same as those of φ; for example, the point $y = h(0) = 0$ is an equilibrium, and if $y > 0$, then $\psi_t(y) \to 0$ as $t \to \infty$, just as

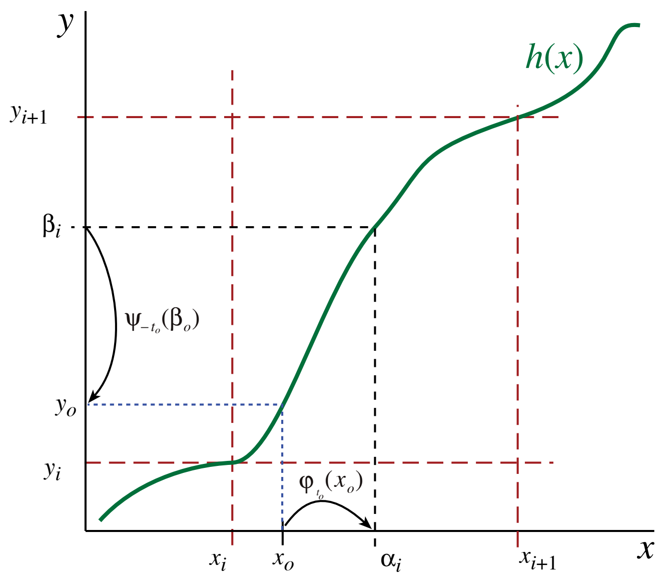

Figure 4.13. *Construction of a homeomorphism for a one-dimensional flow.*

$\varphi_t(x) \to 0$. We used this notion of equivalence in our proof of the theorem in §4.3 that each ODE is equivalent to one with a complete flow. ∎

Two topologically equivalent flows must, in some precise sense, exhibit the same "orbit structure." In particular, for the one-dimensional case, it is quite easy to make a precise statement since the behavior is quite limited.

Theorem 4.10 (One-Dimensional Equivalence). *Two flows φ and ψ in $\mathbb{R}$ are topologically equivalent if and only if their equilibria, ordered on the line, can be put in a one-to-one correspondence, and if and only if the corresponding equilibria have the same topological type (sink, source, or semistable).*

Proof. If a homeomorphism h exists, then to each equilibrium of φ there must be a corresponding equilibrium of ψ and vice versa; thus we can put the equilibria in a one-to-one correspondence. The correspondence is ordered since h is monotone. Conversely, suppose that φ and ψ have corresponding equilibria. We will next explicitly construct h, and show that the flows not only are equivalent but are actually conjugate.[25]

Suppose first, for simplicity, that there are finitely many equilibria. Denote the equilibria of φ by $x_1^* < x_2^* < \cdots < x_n^*$ and of ψ by $y_1^* < y_2^* < \cdots < y_n^*$. It is clear that we must define $h(x_i^*) = y_i^*$. Choose points α_i such that $\alpha_o < x_1^* < \alpha_1 < x_2^* < \cdots < x_n^* < \alpha_n$, and points β_i that are similarly intertwined with y_i^*, as shown in Figure 4.13. We can arbitrarily define $h(\alpha_i) = \beta_i$.To complete the construction of the homeomorphism in an interval between two equilibria $h : (x_i^*, x_{i+1}^*) \to (y_i^*, y_{i+1}^*)$, note that for each $x_o \in (x_i^*, x_{i+1}^*)$, since

[25]The necessity also follows from the Hartman–Grobman theorem.

$\varphi_t(x_o)$ is either monotonically increasing or decreasing with t, there is a *unique* time $t_o \in \mathbb{R}$ such that $\varphi_{t_o}(x_o) = \alpha_i$. As sketched in Figure 4.13, define

$$h(x_o) = y_o = \psi_{-t_o}(\beta_i).$$

This function is a homeomorphism (it is one-to-one since the flow is monotone, and it is continuous and has a continuous inverse since ψ does). Note also that since $\varphi_{t_o-t}\,(\varphi_t(x_o)) = \alpha_i$ we have

$$h(\varphi_t(x_o)) = \psi_{-(t_o-t)}(\beta_i) = \psi_t\left(\psi_{-t_o}(\beta_i)\right) = \psi_t\,(h(x_o))\,,$$

as required. This construction applies in each such interval bounded by two equilibria. We can similarly deal with the two intervals $(-\infty, x_1^*)$ and (x_n^*, ∞). This yields the required homeomorphism on $\mathbb{R}$.

If the number of equilibria is countably infinite, or even uncountably infinite, the analysis is similar. $\square$

Generally, when the dimension of the phase space is larger than one, we must know more than just the number and topological type of the equilibria to determine whether two flows are equivalent; see Exercise 13. We will see such systems in §8.11 when we discuss homoclinic bifurcations.

Sometimes we will not be satisfied by mere topological equivalence—we will want differential properties to be the same. In a previous example we saw that the eigenvalues are not preserved by a topological equivalence (they changed from -1 to -3 at the equilibrium). A notion that does preserve this information is

> $\triangleright$ *diffeomorphic*: Two flows $\varphi_t : A \to A$ and $\psi_t : B \to B$ are diffeomorphic if there is a diffeomorphism h such that $h(\varphi_t(x)) = \psi_t(h(x))$.

We also call two flows *smoothly equivalent* when, in addition to the diffeomorphism h, there is an increasing diffeomorphism $\tau(x, t)$ such that (4.33) is satisfied.

Example: The map $h : \mathbb{R} \to (-1, 1)$ defined by $h(x) = \tanh(x)$ is a diffeomorphism. Applying this to the flow $\varphi_t(x) = xe^{-t}$ gives the new flow $\psi_t = h \circ \varphi_t \circ h^{-1}$, or explicitly

$$\psi_t(y) = \tanh\left(e^{-t}\tanh^{-1}(y)\right).$$

This flow has the vector field

$$\dot{y} = g(y) = \frac{d}{dt}\,\psi_t(y)|_{t=0} = \left(y^2 - 1\right)\tanh^{-1}(y).$$

This ODE has only one equilibrium, $y = 0$, in the interval $(-1, 1)$; since $Dg(0) = -1$, it is stable just like $x = 0$ is for the flow φ. The new flow has equilibria at $y = \pm 1$ as well, but these are not within the space $(-1, 1)$; they correspond to the points $x = \pm\infty$ in the original space. The limiting behavior $\psi_t(y) \underset{t\to-\infty}{\to} \mp 1$, for $y < 0$ and $y > 0$, respectively, reflects the behavior of the original flow, since $\varphi_t(x) \underset{t\to-\infty}{\to} \mp\infty$ for $x < 0$ and $x > 0$, respectively. $\blacksquare$

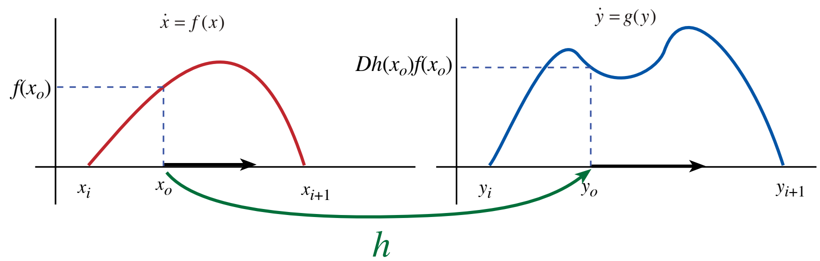

Figure 4.14. *Equivalence between two one-dimensional vector fields,* (4.34).

Although our example on page 131 showed that the flows xe^{-t} and ye^{-3t} are topologically conjugate, we did not show them to be diffeomorphic, since x^3 is not a diffeomorphism. In fact, these two flows cannot be diffeomorphic, as we will see next.

If two flows are diffeomorphic, then the vector fields are related by the derivative of the conjugacy. Suppose that $\dot{x} = f(x)$ generates the flow φ and $\dot{y} = g(y)$ generates ψ. Then

$$\frac{d}{dt}\psi_t(y) = g\left(\psi_t(y)\right) = \frac{d}{dt}h\left(\varphi_t(x)\right) = Dh\left(\varphi_t(x)\right)\frac{d}{dt}\varphi_t(x) = Dh\left(\varphi_t(x)\right) f\left(\varphi_t(x)\right).$$

Setting $t = 0$ in these relations gives a relation between the vector fields:

$$g(y) = g(h(x)) = Dh(x)f(x). \tag{4.34}$$

Equation (4.34), sketched in Figure 4.14, is precisely the result that we would obtain if we simply transform coordinates using the differential equations:

$$y = h(x) \quad \Rightarrow \quad \frac{dy}{dt} = Dh(x)\frac{dx}{dt} = Dh(x)f(x) = g(y).$$

It is easy to see that the eigenvalues of equilibria are preserved by a diffeomorphism. Suppose that x^* is an equilibrium of φ, and $D_x f(x^*) = A$ is the Jacobian matrix. Then upon differentiation of the relation (4.34) by x, we have

$$D_y g(y) D_x h(x) = D_x h(x) D_x f(x) + D_x^2 h(x) f(x).$$

Since h is a diffeomorphism the matrix $H = Dh(x^*)$ is nonsingular, and since $f(x^*) = 0$ at the equilibrium,

$$B \equiv D_y g(y^*) = HAH^{-1}.$$

So the matrices are related by a similarity transformation and therefore have the same eigenvalues (recall Exercise 2.8).

Note in particular that two linear flows can be diffeomorphic only if the fundamental subspaces E^u, E^s, and E^c have the same dimensions; we will see below that this holds more generally. Conversely, two linear ODEs with distinct eigenvalues cannot be diffeomorphic; see Exercise 6. Indeed, two linear flows are diffeomorphic only if their matrices are similar, as shown below.

Theorem 4.11 (Linear Conjugacy). *The flows φ_t and ψ_t of the linear systems $\dot{x} = Ax$ and $\dot{y} = By$ are diffeomorphic if and only if the matrix A is similar to the matrix B.*

Proof. Assume first that A is similar to B, i.e., there is a nonsingular matrix H such that $HA = BH$. The map $h(x) = Hx$ is clearly a diffeomorphism and

$$h(\varphi_t(x)) = He^{tA}x = e^{tHAH^{-1}}Hx = e^{tB}h(x) = \psi_t(h(x)),$$

which implies that the flows φ and ψ are diffeomorphic. Conversely, suppose there is a diffeomorphism g such that $g(\varphi_t(x)) = \psi_t(g(x))$. Setting $g(0) = c$, then $g(\varphi_t(0)) = c = \psi_t(c)$, so c is an equilibrium of ψ. Let $h(x) = g(x) - c$. Then

$$h(\varphi_t(x)) = \psi_t(g(x)) - c = \psi_t(h(x) + c) - c = \psi_t(h(x)). \tag{4.35}$$

Thus $h(x)$ conjugates the flows and fixes the origin. Define the matrix $H = Dh(0)$, and differentiate (4.35) with respect to x, to obtain, at $x = 0$, $He^{tA} = e^{tB}H$. Taking the time derivative of this relation at $t = 0$ yields $HA = BH$, so the matrices are linearly conjugate. □

Example: The matrices

$$A = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}$$

are not similar. Indeed, suppose there were an invertible matrix such that $HA = BH$. Then if $(u, v)^T$ is a column of H, we would have $-2u+v = -2u$ and $-2v = -2v$; consequently, $v = 0$ and $u = c$. Since this is true for each column, H would be singular. However, there does exist a topological conjugacy between the flows $\varphi_t(x) = e^{tA}x$ and $\psi_t(y) = e^{tB}y$. To find $y = h(x) = (h_1(x_1, x_2), h_2(x_1, x_2))$, we first find the flows

$$\begin{aligned} \varphi_t(x_1, x_2) &= \left(e^{-2t}x_1, e^{-2t}x_2\right), \\ \psi_t(y_1, y_2) &= \left(e^{-2t}\left(y_1 + ty_2\right), e^{-2t}y_2\right). \end{aligned}$$

The second component of the conjugacy $h_2(\varphi_t(x)) = \psi_{2t}(y)$ implies

$$h_2\left(e^{-2t}x_1, e^{-2t}x_2\right) = e^{-2t}y_2 = e^{-2t}h_2(x_1, x_2),$$

which has a particular solution $h_2(x_1, x_2) = x_2$. The first component of the conjugacy requires that $h_1\left(e^{-2t}x_1, e^{-2t}x_2\right) = e^{-2t}\left(h_2(x_1, x_2) + tx_2\right)$. To solve this set, $h_1(x_1, x_2) = x_1 + f(x_2)$, to find

$$f(e^{-2t}x_2) = e^{-2t}\left(f(x_2) + tx_2\right).$$

A solution to this is $f(x) = -{}^1\!/\!_2 x \ln|x|$, and if we define $f(0) = 0$, then f is continuous at $x = 0$. Putting this result together with h_1 gives homeomorphism

$$(y_1, y_2) = h(x) = \left(x_1 - \frac{1}{2}x_2 \ln|x_2|, x_2\right);$$

however, h is not a diffeomorphism since its derivative does not exist at the origin. At every other point the vector fields can be transformed using (4.34):

$$\dot{y}_1 = \dot{x}_1 - \frac{1}{2}\dot{x}_2 \ln|x_2| - \frac{1}{2}\dot{x}_2 = -2y_1 + y_2,$$
$$\dot{y}_2 = \dot{x}_2 = -2y_2,$$

showing conjugacy as we expect. ∎

This example can be generalized to show that topological conjugacy of hyperbolic systems depends only on the dimensions of their stable and unstable subspaces: for example a system with complex eigenvalues can be conjugate to one with real eigenvalues; see Exercise 7.

Theorem 4.12. *Suppose A and B are two real, hyperbolic $n \times n$ matrices and $\varphi_t(x) = e^{tA}x$ and $\psi_t(y) = e^{tB}y$ the corresponding flows. Then φ and ψ are topologically conjugate if and only if the dimensions of the stable and unstable spaces of A are equal to the corresponding dimensions for B.*

Sketch of Proof. The necessity of this condition is easy to see. Any homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ must map bounded sets to bounded sets. Moreover, for any $x \in E^s_A$, we have $\lim_{t\to\infty} \varphi_t(x) = 0$; consequently, since h is continuous $\lim_{t\to\infty} h(\varphi_t(x)) = h(0) = \lim_{t\to\infty} \psi_t(h(x))$. Since $h(0)$ is bounded, then $y = h(x)$ must be in E^s_B, and indeed $h(0) = 0$ because every orbit in E^s_B approaches the origin. Consequently, $h : E^s_A \to E^s_B$ is a homeomorphism, which implies that these spaces must have the same dimension. The same can be said for the unstable spaces.

The proof of the converse requires a bit more work: given that the dimensions of the stable and unstable spaces are the same we must construct the conjugacy. Since the stable spaces E^s_A and E^s_B are invariant under the flows, we start by constructing a map $h_s : E^s_A \to E^s_B$. A similar map h_u can be constructed for the unstable spaces. In the end, we write any vector $x = \pi_u(x) + \pi_s(x)$, where π_i are projection operators onto the unstable and stable spaces of A, respectively, and the full conjugacy is then $h(x) = h_s(\pi_s(x)) + h_u(\pi_u(x))$.

The proof is simple for the case when A and B are semisimple and all their eigenvalues are real. Then both A and B are linearly conjugate to real diagonal matrices and so to the systems $\dot{x}_i = \lambda_i x_i$ and $\dot{y}_i = \mu_i y_i$. Order the eigenvalues so that $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_k \geq 0 > \lambda_{k+1} \geq \cdots \geq \lambda_n$ and similarly for μ_i. By our previous argument the number, k, of positive eigenvalues must be the same. Now we construct conjugacies for each one-dimensional system, mapping λ_i to μ_i, by choosing

$$h_i(x_i) = \operatorname{sgn}(x_i)\,|x|^{a_i}\,, \text{ where } a_i = \mu_i \big/ \lambda_i\,.$$

Then $h_i(e^{\lambda_i t}x_i) = e^{\mu_i t}\operatorname{sgn}(x_i)\,|x_i|^{a_i} = e^{\mu_i t}h_i(x_i)$. Whenever $\lambda_i \neq \mu_i$, h_i is *not* a diffeomorphism. Note also that we cannot get out of this difficulty by relaxing the conjugacy requirement to one of equivalency, since the ratio of the eigenvalues may be different for different i, and thus we would need a different time scaling for each dimension.

In the general case we construct h_s by first finding norms that are adapted to the matrices A and B. These norms are constructed so that $\|e^{tA}\pi_s(x)\|_A \leq e^{-\alpha t}\|\pi_s(x)\|_A$ for

$t \geq 0$, i.e., eliminating the constant K in (4.20). The point of these norms is that each trajectory crosses its respective unit sphere $\|x\| = 1$ exactly once. The unit spheres in the new norms are then used to define h_s as the "identity" map from the A-sphere to the B-sphere. The homeomorphism is extended from the spheres by flowing, just like we did for the one-dimensional case. The full proof is given, for example, in (Robinson 1999, see §4.7). □

4.8 Hartman–Grobman Theorem

We showed in §4.7 that linear, hyperbolic systems come in a few equivalence classes, categorized solely by the dimension of their stable and unstable spaces. Now we show that nonlinear systems sometimes "look like" their linearizations near hyperbolic equilibria. The formal statement of this result was proved independently by Hartman in 1960 and Grobman in 1959.

Theorem 4.13 (Hartman–Grobman). *Let x^* be a hyperbolic equilibrium point of a C^1 vector field $f(x)$ with flow $\varphi_t(x)$. Then there is a neighborhood N of x^* such that φ is topologically conjugate to its linearization on N.*

It is interesting to note that while the theorem requires a smooth ODE, it does not say that the flow is diffeomorphic to its linearization. A theorem due to Sternberg does provide a diffeomorphism; however, it requires an additional hypothesis: the eigenvalues must satisfy a "nonresonance" condition (Sternberg 1958).

Note that the Hartman–Grobman theorem requires that the equilibrium be hyperbolic. As we shall see in Chapter 6, the topological classification of nonhyperbolic equilibria will depend upon more than just the linearization of the system.

Discussion of Proof. The construction of the homeomorphism is rather clever and potentially useful, so we sketch it here. As is now usual, we begin with an ODE of the form

$$\dot{x} = Ax + g(x),$$

where A is a hyperbolic matrix, and the term $g \in C^1$ represents the nonlinear term, so that $g = o(x)$. Define also the flow of the linear equation $\psi_t(x) = e^{tA}x$. Since the theorem is to be proved only locally, we can modify the ODE by defining a new nonlinearity $\tilde{g}$ such that $\tilde{g}(x) = g(x)$ for some neighborhood N of 0, and $\tilde{g}(x) = 0$ for x outside some larger neighborhood M. This can be done so that $\tilde{g}$ is still a smooth function. Moreover, $\tilde{g}$ is bounded, since it vanishes outside a compact set. Let φ_t be the flow for the modified ODE. This flow agrees with the linear flow while the orbit stays outside M. (See the following examples to understand why this modification is needed.)

Our goal is to find a homeomorphism h satisfying

$$\psi_t\left(h(x)\right) = h\left(\varphi_t(x)\right), \text{ that is, } h(x) = e^{-tA} \circ h \circ \varphi_t(x). \tag{4.36}$$

Suppose first that H is a homeomorphism that satisfies (4.36) for one value of time, say, $t = 1$, e.g.,

$$H_1(x) = e^{-A} H_1(\varphi_1(x)). \tag{4.37}$$

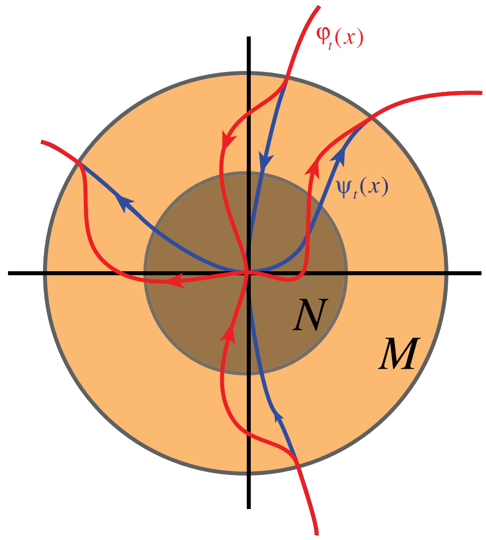

Figure 4.15. *The homeomorphism* (4.36).

In addition, suppose we can show that H_1 is the unique such homeomorphism (among the class of continuous functions such that $H_1 - id$ is bounded). Now let

$$H_t(x) = e^{-tA} \circ H_1 \circ \varphi_t(x);$$

a sketch of this relation is shown in Figure 4.15. We then claim that H_t is also a homeomorphism that satisfies (4.37). This follows from the group property of the flow φ:

$$\begin{aligned} e^{-A} \circ H_t \circ \varphi_1(x) &= e^{-A} \circ e^{-tA} \circ H_1 \circ \varphi_t \circ \varphi_1(x) \\ &= e^{-tA} \circ e^{-A} \circ H_1 \circ \varphi_1 \circ \varphi_t(x) \\ &= e^{-tA} \circ H_1 \circ \varphi_t(x) = H_t(x). \end{aligned}$$

Consequently, H_t satisfies (4.37); however, since we asserted that H_1 is the unique such homeomorphism, we must have $H_t = H_1$. Therefore,

$$H_1 = e^{-tA} \cdot H_1 \cdot \varphi_t(x).$$

So H_1 is also the homeomorphism for any time t! This can be seen as well by considering the following diagram:

$$\begin{array}{ccccc} x & \xrightarrow{\varphi_t} & x(t) & \xrightarrow{\varphi_{1-t}} & x(1) \\ H_1 \downarrow & & H_1 \downarrow & & H_1 \downarrow \\ y & \xrightarrow{e^{tA}} & y(t) & \xrightarrow{e^{(1-t)A}} & y(1) \end{array} .$$

We have just shown that this diagram commutes; that is, if we go from x to $y(1)$ by any path we obtain the same result.

So we reduce the problem to solving for H_1, the conjugacy at $t = 1$. Basically, we can do this iteratively by starting with the assumption that $H_1^{(0)}(x) = x$, and defining

$$H_1^{(i+1)}(x) = e^{-A} \circ H_1^{(i)} \circ \varphi_1(x), \;\; i = 0, 1, \ldots . \tag{4.38}$$

The theorem actually proves that there is a neighborhood of the origin for which (a version of) this iteration converges and that H_1 is unique among all homeomorphisms that are near the identity.

The full proof of the theorem is in (Robinson 1999, §5.7). □

Example: The simple two-dimensional system

$$\dot{x} = x,$$
$$\dot{y} = -y + x^2$$

has a saddle equilibrium at the origin. The linear matrix for the saddle is$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, which is conveniently diagonal, so that $e^{tA}\,(x, y)^T = \left(e^t x, e^{-t} y\right)^T$. The nonlinear system can be easily solved analytically to obtain the flow

$$\varphi_t(x, y) = \begin{pmatrix} e^t x \\ e^{-t} y + \frac{1}{3}\left(e^{2t} - e^{-t}\right) x^2 \end{pmatrix}.$$

As a consequence, the homeomorphism $H = H_1$ in (4.37) must satisfy the equation

$$H(x, y) = e^{-A} H(\varphi_1(x, y)) = \begin{pmatrix} e^{-1} & 0 \\ 0 & e \end{pmatrix} H(ex, e^{-1} y + kx^2), \tag{4.39}$$

where $k = \frac{e^3 - 1}{3e}$. It is convenient to solve for the two components of H separately; let $H = (K, L)^T$. Then the iterative equation (4.38) for K is

$$K^{(i+1)}(x, y) = \frac{1}{e} K^{(i)} \left(ex, \frac{1}{e} y + kx^2\right).$$

The superscripts on this equation indicate that we will attempt to solve it iteratively. Starting with $K^{(0)}(x, y) = x$, the identity, then $K^{(1)} = \frac{1}{e}(ex) = x$; thus, $K(x, y) = x$ is the solution. The formal iterative equation for L from (4.39) is

$$L^{(i+1)}(x, y) = e L^{(i)} \left(ex, \frac{1}{e} y + kx^2\right),$$

which looks like it should be amenable to iteration in the same way. However, because there is a factor of e in front of the right-hand side, we cannot iterate this equation in the form as written—the result will diverge (try it and see!). Instead we must invert it. To do this, set $\xi = ex$ and $\eta = y/e + kx^2$, so that $x = \xi/e$, and $y = e\left(\eta - k\xi^2/e^2\right)$. Using this to invert the equation above and write it as an iteration yields

$$L^{(i+1)}(x, y) = \frac{1}{e} L^{(i)} \left(\frac{1}{e} x, ey - \frac{k}{e} x^2\right).$$

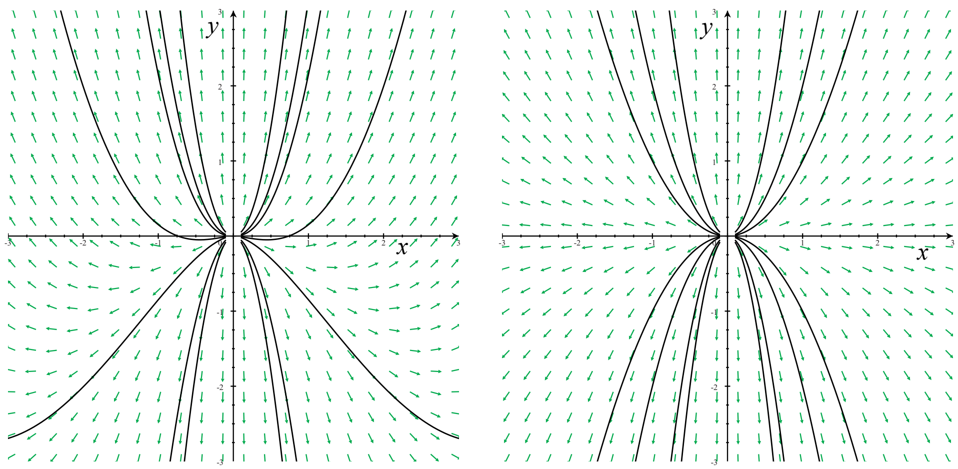

Figure 4.16. *Phase planes for the nonlinear flow (left) and linear flow (right) in* (4.40). *The constructed homeomorphism maps the two families of curves onto each other.*

As before we start the iteration with the identity, $L^{(0)}(x, y) = y$, and now obtain

$$L^{(1)}(x, y) = \frac{1}{e} L^{(0)} \left(\frac{1}{e} x, ey - \frac{k}{e} x^2 \right) = \frac{1}{e} \left(ey - \frac{k}{e} x^2 \right) = y - ke^{-2} x^2,$$

$$L^{(2)}(x, y) = \frac{1}{e} \left(ey - \frac{k}{e} x^2 - ke^{-2} \left(\frac{x}{e} \right)^2 \right) = y - ke^{-2}(1 + e^{-3}) x^2,$$

$$L^{(3)}(x, y) = \frac{1}{e} \left(ey - \frac{k}{e} x^2 - ke^{-2}(1 + e^{-3}) \left(\frac{x}{e} \right)^2 \right) = y - ke^{-2}(1 + e^{-3} + e^{-6}) x^2.$$

This series limits to

$$L(x, y) = y - ke^{-2}(1 + e^{-3} + e^{-6} + e^{-9} + \cdots) x^2 = y - \frac{ke^{-2}}{1 - e^{-3}} x^2 = y - \frac{1}{3} x^2.$$

So the actual homeomorphism is $H(x, y) = \left(x, y - x^2/3\right)$. The reader is encouraged to verify that this actually works by doing the calculation (4.36). ∎

Example: The homeomorphism for the Hartman–Grobman theorem is guaranteed to exist only in a neighborhood of the origin. We can see that this is the case if we consider the ODEs

$$\dot{x} = 2x,$$
$$\dot{y} = 4y + x^2,$$

which have a source at the origin. The flow for this system and its linearization are

$$\varphi_t(x, y) = \begin{pmatrix} e^{2t} x \\ e^{4t}(y + tx^2) \end{pmatrix}, \qquad e^{tA} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^{2t} x \\ e^{4t} y \end{pmatrix}. \tag{4.40}$$

These flows are shown in Figure 4.16. If we attempt the calculation as we did in the previous example, we will find that $H(x, y) = (x, y + g(x, y))$ but that the iteration for g does not converge. Instead of doing this, we modify the vector field:

$$\begin{aligned}\dot{x} &= 2x,\\ \dot{y} &= 4y + b(x^2),\end{aligned}$$

where the function b is a "bump" function. That is, we want $b(\xi) = \xi$ for small ξ and for it to vanish for large ξ. So we set

$$b(\xi) = \begin{cases} \xi, & |\xi| < \varepsilon, \\ 0, & |\xi| > \delta, \end{cases}$$

for some arbitrarily chosen $0 < \varepsilon < \delta$. We assume that b connects these two values smoothly.[26] The new vector field has a flow identical to the original nonlinear one when $x^2 < \varepsilon$ but is identical to the linear flow when $x^2 > \delta$. The fact that the Hartman–Grobman theorem is only locally valid is made manifest by this modification.

When we integrate the modified equations, we obtain $x(t) = e^{2t}x_o$ and

$$y(t) = e^{4t}\left(y_o + \int_0^t e^{-4s} b(e^{4s}x_o^2)ds\right) \equiv e^{4t}\left(y_o + B(x_o^2, t)\right).$$

The new function $B(x^2, t)$ cannot be obtained explicitly—especially since we have not explicitly specified b! However, we do know that if $x^2(s) < \varepsilon$ for all $0 < s < t$, i.e., if $|x_o| < \sqrt{\varepsilon}e^{-2t}$, then $b(x^2(s)) = x^2(s)$ along the entire integration path and we obtain $B(x_o^2, t) = tx_o^2$. Similarly, if $x^2(s) > \delta$ for all $0 < s < t$, i.e., if $|x_o| > \sqrt{\delta}$, then $b(x^2(s)) = 0$, so that $B(x_o^2, t) = 0$. Setting $t = 1$, and letting $B(x^2) = B(x^2, 1)$, we have

$$B(x^2) = \begin{cases} x^2, & |x| < \sqrt{\varepsilon}e^{-2}, \\ 0, & |x| > \sqrt{\delta}. \end{cases}$$

Putting the new flow into (4.37), we obtain the equation for H:

$$H(x, y) = e^{-A}H(\varphi_1(x, y)) = \begin{pmatrix} e^{-2} & 0 \\ 0 & e^{-4} \end{pmatrix} H\left(e^2x, e^4(y + B(x^2))\right).$$

As before we write $H = (K, L)^T$. The equation for K has the simple solution $K(x, y) = x$. For L we obtain

$$L(x, y) = e^{-4}L\left(e^2x, e^4(y + B(x^2))\right).$$

Iterating this starting with $L^{(0)}(x, y) = y$, we get

$$\begin{aligned}L^{(1)}(x, y) &= y + B(x^2),\\ L^{(2)}(x, y) &= y + B(x^2) + e^{-4}B(e^4x^2),\\ L^{(3)}(x, y) &= y + B(x^2) + e^{-4}B(e^4x^2) + e^{-8}B(e^8x^2).\end{aligned}$$

After N steps this gives the obvious result

$$L^{(N)}(x, y) = y + \sum_{n=0}^{N-1} e^{-4n}B(e^{4n}x^2).$$

[26] It is a standard trick in analysis that such "bump" functions can be made arbitrarily smooth, and even C^∞; see, for example, (Friedman 1982, Problem 3.3.1).

Note that if we set $B(x^2) = x^2$, then this series sums to Nx^2, which does not converge as $N \to \infty$. However, since B vanishes when its argument is large, then the series actually terminates after finitely many terms. Explicitly, choose an N such that $e^{4N}x^2 \geq \delta$, or $N(x) \geq \frac{1}{4}\ln(\delta/x^2)$, then $B(e^{4N}x^2) = 0$, so that this term and all the following ones vanish. Using this we can "take the limit" to obtain

$$L(x, y) = y + \sum_{n=0}^{N(x)} e^{-4n} B(e^{4n}x^2).$$

Since the sum is finite, it is convergent. This is the local homeomorphism guaranteed by the theorem. Note that it is not unique because we have considerable freedom in choosing b; however, once we have chosen the function $b(x)$, we get a unique homeomorphism. ∎

The Hartman–Grobman theorem implies Theorem 4.6: if x^* is a hyperbolic equilibrium point with $\mathrm{Re}(\lambda) < 0$, then since the linear system is asymptotically stable, so is the nonlinear system.

The Hartman–Grobman theorem says nothing about the structure of the motion in the neighborhood of a nonhyperbolic equilibrium. This case is considerably more intricate—we will discuss it in Chapter 6 and Chapter 8.

4.9 Omega-Limit Sets

We now develop some terminology that will help in the classification of orbits. Since—as we saw in §4.3—up to reparameterization of time, ODEs give rise to complete flows, we now consider a general flow, $\varphi_t(x)$. Our goal is to study properties of the orbits,

$$\Gamma_x = \{\varphi_t(x) : t \in \mathbb{R}\}. \tag{4.41}$$

In some cases, as in (4.2), we will consider just the forward orbit of x, the set

$$\Gamma_x^+ = \left\{\varphi_t(x) : t \in \mathbb{R}^+\right\}, \tag{4.42}$$

or the similarly defined backward orbit, Γ_x^-. One of the main goals of theory of dynamical systems is to give a geometrical classification of the types of orbits that occur in a given flow.

One important characterization of orbits is their "ultimate" or asymptotic behavior as $t \to \infty$, if this exists in some sense. Asymptotic behavior is defined in terms of *limit points*; recall §3.1: a point y is a *limit point* of the forward orbit of x if there is a sequence $t_1 < t_2 < \cdots < t_k \ldots$ such that $t_k \to \infty$ and $\varphi_{t_k}(x) \to y$ as $k \to \infty$. The asymptotic behavior of an orbit is its

▹ *omega-limit set*: The collection of all limit points of Γ_x^+ is the *omega-limit set* of x, denoted $\omega(x)$.

It is easy to see from the definition that if $z \in \Gamma_x$, $\omega(z) = \omega(x)$. Thus instead of $\omega(x)$, we can just as well write $\omega(\Gamma_x)$, the ω-limit set of the entire trajectory. Similarly, we can define a limit set for $t \to -\infty$:

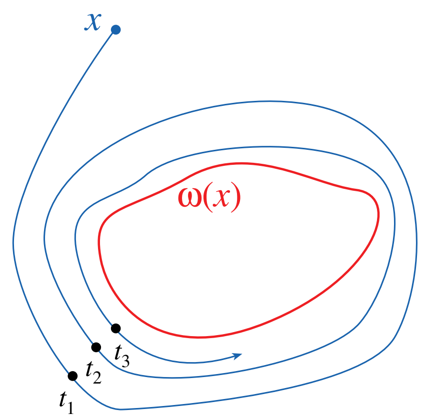

Figure 4.17. *The omega-limit set can be a limit cycle.*

▷ *alpha-limit set*: $\alpha(x)$ is the collection of all limit points of Γ_x^- .

A simple example of an ω-limit set is an asymptotically stable equilibrium, another example is a periodic orbit that attracts a trajectory; see Figure 4.17. Such an orbit is called a

▷ *limit cycle*: A periodic orbit γ that is the omega or alpha-limit set of a point $x \notin \gamma$ is a limit cycle.

Thus, a limit cycle is an invariant loop with the property that there is a nearby orbit that spirals either toward it or away from it.[27] As we will see in Chapter 6, limit cycles are common for planar flows and more generally can arise through a "bifurcation" of an equilibrium when it becomes unstable; see §8.8.

Example: The planar system

$$\begin{aligned} \dot{x} &= x(1-r^2) - y, \\ \dot{y} &= y(1-r^2) + x \end{aligned} \tag{4.43}$$

is most easily analyzed in polar coordinates. The radial equation is

$$\dot{r} = \frac{1}{r}(x\dot{x} + y\dot{y}) = r(1-r^2). \tag{4.44}$$

This one-dimensional system has a source at $r = 0$ and a sink at $r = 1$ (negative values of r are not allowed). The dynamics of $\theta = \tan^{-1}(y/x)$ are given by

$$\dot{\theta} = \frac{1}{r^2}(x\dot{y} - y\dot{x}) = \frac{1}{r^2}(x^2 + y^2) = 1.$$

[27] Sometimes limit cycles are defined as isolated periodic orbits. This definition is not equivalent to ours, as a periodic orbit in a planar system could bound a disk of other periodic orbits and still be the limit of a spiraling trajectory from the outside.

Thus the dynamics on the circle $\gamma = \{(r, \theta) : r = 1\}$ are simply $\theta(t) = \theta_o + t$: it is a periodic orbit. The orbit γ is an asymptotically stable limit cycle because the radial equation shows that $r(t) \to 1$ for any $r(0) \neq 0$. ∎

Note that a limit cycle is closed (the loop γ includes all of its limit points) and invariant, $\varphi_t(\gamma) = \gamma$. These properties are generally true for ω-limit sets, as we will see in the following three fundamental lemmas that define the basic structure of ω-limit sets.

Lemma 4.14 (Closure). *$\omega(x) = \bigcap_{T \geq 0} \bar{\Gamma}^+_{\varphi_T(x)}$, where $\bar{\Gamma}^+_x$ is the closure of the forward orbit of x. Hence, $\omega(x)$ is closed.*

Proof. If $z \in \omega(x)$, then $z \in \bar{\Gamma}^+_{\varphi_T(x)} = cl\{y : y = \varphi_t(x),\ t \geq T\}$ for any T, since this includes all limit points. Therefore, z is in the intersection of these sets. This proves that $\omega(x) \subset \bigcap_{T\geq 0} \bar{\Gamma}^+_{\varphi_T(x)}$. Now suppose that $z \in \bigcap_{T \geq 0} \bar{\Gamma}^+_{\varphi_T(x)}$, or equivalently for any T, $z \in \bar{\Gamma}^+_{\varphi_T(x)}$. If there is a time t such that $z = \varphi_t(x)$, then there must be a larger time for which this is true as well; this implies that z must appear infinitely often in the orbit Γ^+_x, and so $z \in \omega(x)$. Otherwise z is in the closure of Γ^+_x but is not in the orbit itself, and by definition of "closure," it is a limit point of the orbit. Finally, recall that the intersection of a family of closed sets is closed. □

Lemma 4.15 (Invariance). *The ω-limit set is invariant.*

Proof. If $y \in \omega(x)$, then there is a sequence t_k such that $\varphi_{t_k}(x) \to y$. Continuity then implies that for any fixed $s \in \mathbb{R}$, $\varphi_{t_k+s}(x) \to \varphi_s(y)$. Therefore, $\varphi_s(y) \in \omega(s)$. □

Now suppose that there is a metric $\rho(x, y)$ defined on the phase space; recall §3.2. We define the distance between a point, x, and a set, S, by

$$\rho(x, S) = \inf_{y \in S} \rho(x, y).$$

We will show next that when an orbit of a flow is bounded, it must approach its ω-limit set, in the sense that $\rho(\varphi_t(x), \omega(x)) \to 0$; in this case we say that $\varphi_t(x) \to \omega(x)$. We will also show that in this case that $\omega(x)$ is

▷ *connected*: A set S is connected if it *cannot* be partitioned into two nonempty sets such that each subset has no points in common with the closure of the other.

Thus, $\mathbb{R}^+$ is connected; for example, it can be partitioned into $A = (0, 1)$, and $B = [1, \infty)$, but $\bar{A} \cap B = \{1\}$ is not empty.

Lemma 4.16 (Compact and Connected). *If the forward orbit of x is contained in a compact set, then $\omega(x)$ is nonempty, compact, and connected. Furthermore, $\varphi_t(x) \to \omega(x)$ as $t \to \infty$.*

Proof. The sets $\bar{\Gamma}^+_{\varphi_T(x)} = cl\{\varphi_t(x) : t \geq T\}$ are nested, since $\bar{\Gamma}^+_{\varphi_{T+s}(x)} \subset \bar{\Gamma}^+_{\varphi_T(x)}$ for any $s > 0$. Since, by assumption, the forward orbit of x is contained in a compact set, each

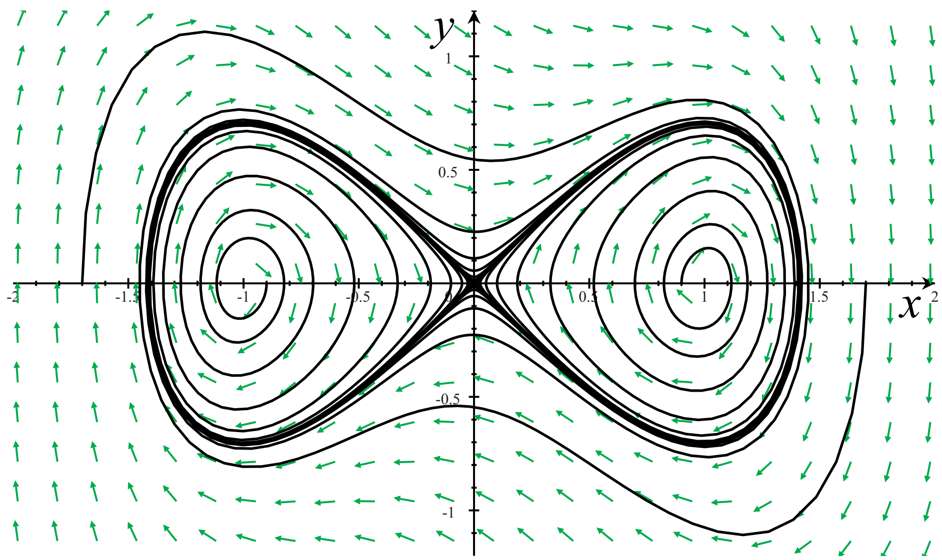

Figure 4.18. *Attracting figure-eight orbit of* (4.45) *for* $\mu = 0.5$.

$\bar{\Gamma}^{+}_{\varphi_T(x)}$ is also compact. According to Lemma 4.14, $\omega(x)$ is the intersection of these sets, and the intersection of a collection of nested closed sets is nonempty; then $\omega(x)$ is nonempty. Moreover, since $\omega(x)$ is closed and contained in a compact set, it is compact.

Now suppose that $\omega(x)$ is not connected, i.e., that there are two disjoint, closed components A and B such that $\omega(x) = A \cup B$. By definition, for any $z_A \in A$ there is a sequence of times t_{k_A} for which $\varphi_{t_{k_A}}(x) \to z_A$. Similarly for $z_B \in B$. Since for each sequence $t_k \to \infty$, there are infinitely many neighboring times for which $t_{k_A} < t_{k_B} < t_{k_A+1}$ and so the orbit segment $\{\varphi_t(x) : t_{k_A} \le t \le t_{k_B}\}$ connects points arbitrarily close to z_A to points arbitrarily close to z_B. Since $\omega(x)$ is closed, it contains the limits of these segments and therefore cannot be disconnected. (More generally, any intersection of a nested collection of compact, connected sets is connected.)

Finally, suppose that $\rho(\varphi_t(x), \omega(x))$ does not go to zero. Then there must be a subsequence $\varphi_{t_k}(x)$ of points that stay a distance δ away from $\omega(x)$. However, since this sequence is contained in a compact set, it has a convergent subsequence, which would be a limit point not in $\omega(x)$, but this is a contradiction. In conclusion, $\rho(\varphi_t(x), \omega(x)) \to 0$. $\square$

Example: Consider the system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x - x^3 - \mu y \left(y^2 - x^2 + \frac{1}{2}x^4 \right). \end{aligned} \tag{4.45}$$

When $\mu = 0$, (4.45) is a Hamiltonian system with $H = {}^1\!/\!_2(y^2 - x^2 + {}^1\!/\!_2 x^4)$. The level set $H = 0$ is a figure eight, with $H < 0$ inside its lobes and $H > 0$ outside; see Figure 4.18. The

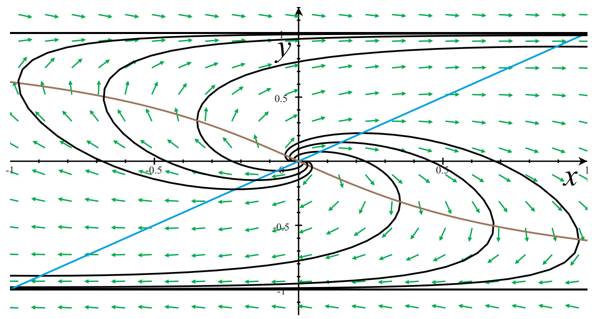

Figure 4.19. *Phase portrait of the system* (4.46)*, showing the nullclines (blue and brown).*

term proportional to μ in the y-equation is specially chosen so that it vanishes on $H = 0$. Thus, any orbit that starts on this curve will stay on it even when $\mu \neq 0$. Note that the rate of change of energy is given by

$$\frac{dH}{dt} = \frac{\partial H}{\partial x} y + \frac{\partial H}{\partial y}\left(x - x^3 - 2\mu y H\right) = -2\mu y^2 H.$$

Consequently, when $y \neq 0$ and $H < 0$ (inside the lobes of the figure eight), H is increasing and when $H > 0$ (outside the figure eight), H is decreasing. Therefore, trajectories move toward the figure eight contour except possibly when $y = 0$. Only the points $(0, 0)$ and $(\pm 1, 0)$ on this set are invariant, so we can conclude, using LaSalle's invariance principle, Theorem 4.9, that $|H(x(t), y(t))|$ monotonically decreases to zero as $t \to \infty$ for every point except the equilibria $(\pm 1, 0)$.

We can, therefore, completely characterize the ω-limit sets for each point in the plane. A point x inside the right lobe of the figure eight (but not at the equilibrium $(-1, 0)$) has an ω-limit set given by the entire right lobe—each point on the lobe is a limit point of its trajectory. A similar discussion applies to points inside the left lobe. Any point outside the two lobes (i.e., with $H > 0$) has the entire figure eight as its ω-limit set. The ω-limit set of any point on the figure eight is the origin. Finally, each equilibrium is its own ω-limit set. ∎

If $\omega(x)$ is not compact, then it need not be connected.

Example: Consider the system

$$\begin{aligned} \dot{x} &= y + x(1 - y^2), \\ \dot{y} &= (1 - y^2)(y - x). \end{aligned} \tag{4.46}$$

There is a spiral source at the origin, and the lines $y = \pm 1$ are invariant. Let $R = \{(x, y) \neq (0, 0) : |y| < 1\}$ be the open region that is bounded by these lines. A numerical phase portrait, see Figure 4.19, shows that trajectories starting in R spiral outward and approach either $y = +1$ or $y = -1$. However, they appear to continually spiral and never settle down on either line. In particular, when the trajectory crosses the nullcline $N_y = \{y = x\}$, then $\dot{y}$ changes sign: in particular if $y > 0$ and is approaching 1, then it will cross this line and begin to diverge from 1. Thus for any point $z \in R$, it appears that $\omega(z) = \{y = 1\} \cup \{y = -1\}$, which is not connected. The conclusion can be made rigorous by consideration of the global phase portrait; see Exercise 6.14. ∎

There are two other characterizations of long-time behavior that are of interest:

▷ *nonwandering*: A point x is nonwandering if for *every* neighborhood W of x and every time $T > 0$ there is time $t > T$ such that $\varphi_t(W) \cap W \neq \emptyset$.

In other words, a nonwandering point has nearby points that continually return. Consequently, any periodic orbit is nonwandering. Moreover, it can be shown that every point in an ω-limit set is nonwandering; see Exercise 14.

▷ *minimal set*: A set S is minimal if it is closed, nonempty, and invariant and does not contain any such set as a proper subset.

For example, a periodic orbit is minimal, but the union of two periodic orbits is not.

Theorem 4.17. *Suppose S is compact; then S is minimal if and only if for each $x \in S$ we have $S = \omega(x)$.*

Proof. First assume that $S = \omega(x)$ but is not minimal. Then there is a closed set $B \subset S$ that is invariant. However, if $x \in B$, then $\omega(x) \in B$. This is a contradiction, so S must be minimal. Now assume that S is minimal, but there is an $x \in S$ for which $\omega(x) \neq S$. Since S is compact, so is $\omega(x)$, and Lemma 4.14 implies that $\omega(x)$ is invariant, so S has an invariant subset. Again, this is a contradiction. □

4.10 Attractors and Basins

Informally, an attractor is an invariant set toward which all nearby trajectories move. We saw in §4.5 that any equilibrium that is linearly asymptotically stable satisfies this condition. Our goal is to define the notion of attractor without reference to the kind of orbit or orbits that it contains; indeed, some attractors consist of infinitely many orbits. We start by generalizing the definition stability that we used for equilibria in §4.5 to arbitrary invariant sets (recall the definition of invariant set in §4.1):

▷ *stability*: An invariant set Λ is stable if for any neighborhood N of Λ there is a subset M of N such that all points that start in M stay in N for all $t > 0$.

▷ *asymptotic stability*: An invariant set Λ is asymptotically stable if it is stable and there is a neighborhood N such that for each $x \in N$, $\rho(\varphi_t(x), \Lambda) \to 0$ as $t \to \infty$.

Since these definitions always refer to a neighborhood of the invariant set, we will define an attractor by constructing a special neighborhood that will envelope it:

▷ *trapping region*. A set N is a trapping region if it is compact and $\varphi_t(N) \subset \text{int}(N)$ for $t > 0$.

Here, "int(N)" denotes the "interior" of the set N. Thus, a trapping region is strictly "forward invariant." Note also that $\varphi_{t+s}(N) = \varphi_s(\varphi_t(N)) \subset \text{int}(\varphi_t(N)) \subset \text{int}(N)$ for any $s, t > 0$; thus the sequence of sets $\varphi_{t_i}(N)$ is nested for any increasing sequence t_i.

Trapping regions are computationally and analytically quite easy to find: it is sufficient that the vector field point inward everywhere on the boundary. The maximal invariant set inside a trapping set is called an

▷ *attracting set*: A set Λ is an attracting set if there is a compact trapping region $N \supset \Lambda$ so that

$$\Lambda = \bigcap_{t>0} \varphi_t(N). \tag{4.47}$$

Note that since the collection $\{\varphi_t(N) : t \geq 0\}$ is a set of closed and nested sets, the intersection, Λ, is closed and nonempty. For compact sets there is no difference between the concepts of asymptotic stability and attracting set.

Lemma 4.18. *An attracting set is asymptotically stable. Conversely, if a compact set is asymptotically stable, then it is an attracting set.*

Proof. First, suppose Λ is an attracting set; then by definition every point in any trapping region N stays in N, so Λ is stable, and approaches Λ—so it is asymptotically stable.

Conversely, assume that A is compact and asymptotically stable. To show it is an attracting set we must construct a trapping set. Since A is asymptotically stable, there is a neighborhood U of A for which all points approach A and stay in some larger neighborhood D. Since A is compact, a compact subset of U can be chosen if needed. Now we have to find a subset of U that is forward invariant. Since all points $x \in U$ eventually approach A, there exists a time $T(x)$ for each $x \in U$ such that $\varphi_t(x) \in U$ for all $t > T(x)$. Moreover, since U is compact, the function $T(x)$ has a maximum:

$$T_{\max} = \max_{x \in U} (T(x)) .$$

Therefore, $N = \varphi_{T_{\max}}(U) \subset U$. By construction $\varphi_t(N) \subset \text{int}(N)$, so N is a trapping region for A. □

Any attracting set has a maximal trapping region that is called the *stable set* of Λ or the

▷ *basin of attraction*, $W^s(\Lambda)$: The basin (or stable set) of an invariant set Λ is the set of all points x for which $\rho(\varphi_t(x), \Lambda) \to 0$ as $t \to \infty$.

Thus if Λ is an attracting set with trapping region N, then

$$W^s(\Lambda) = \bigcup_{t \leq 0} \varphi_t(N).$$

Note that the definition of asymptotic stability is equivalent to the fact that Λ is stable and $\Lambda \subset \mathrm{int}(W^s(\Lambda))$. This concept also provides another way of stating Lemma 4.16: if the forward orbit of x is contained in a compact set, then $x \in W^s(\omega(x))$.

Example: Consider a diagonalizable linear system with a matrix A whose eigenvalues are all negative. The system can be put in diagonal form by a linear coordinate transformation to obtain $\dot{x}_j = \lambda_j x_j$. The unit square $N = \{x : |x_j| \le 1\}$ is mapped to the set $\varphi_t(N) = \left\{x : |x_j| \le e^{\lambda_j t}\right\} \subset \mathrm{int}(N)$ when $t > 0$, so N is a trapping region. Moreover, the origin is an attractor and the entire phase space is the basin of the origin: $W^s(\{0\}) = \mathbb{R}^n$. ∎

Following Charles Conley, an attractor is an attracting set with an additional assumption of "irreducibility" (Ruelle 1981). Basically, we would like to decompose attracting sets into their fundamental components. There are several possible requirements that one could add to our definition; for example, an attractor could be minimal (Perko 2000), "chain transitive" (Robinson 1999), or contain a dense orbit (Guckenheimer and Holmes 1983). We follow (Field 1996) to define an

▷ *attractor*: A set Λ is an attractor if it is an attracting set and there is some point x such that $\Lambda = \omega(x)$.

Example: Consider the system

$$\dot{x} = x(1 - x^2),$$
$$\dot{y} = -y.$$

There are three equilibria (0, 0) (a saddle), and $(\pm 1, 0)$ (sinks). The set $\Lambda = \{-1 \le x \le 1, y = 0\}$ is, by our definition, an attracting set. Its basin is the entire plane. For the trapping set we could take any rectangular disk enclosing Λ. Note that there is no orbit, however, that approaches all the points in Λ; indeed, almost every trajectory approaches one of the two sinks. Thus the only attractors for this example are the equilibria $(\pm 1, 0)$. ∎

The definition of attractor that we give follows the school of Conley (Conley 1978; Easton 1998). A related concept, a *measure attractor*, is due to John Milnor: it is a set that attracts a set of positive measure but does not necessarily have an attracting neighborhood (Milnor 1985a, b). There are interesting examples of sets that attract many but not all points in a neighborhood, and even sets whose basin is *nowhere dense* (Alexander et al. 1996). We will always assume that an attractor has an attracting neighborhood.

Example: In §4.6 it was shown that the Lorenz system (4.26) has a Lyapunov function about the origin when $\sigma > 0, b > 0$, and $r < 1$. Lorenz studied the system at much different values: $\sigma = 10$, $b = 8/3$, and $r = 28$. Here, it has an attracting set that appears to be a "strange" set: a fractal.[28] We can demonstrate that this system does have an attractor, when $\sigma, b > 0$, by constructing a trapping region. Consider the ball

$$B_R = \left\{(x, y, z) : x^2 + y^2 + (z - r - \sigma)^2 \le R^2\right\}. \tag{4.48}$$

[28] We will discuss strange sets in §7.3.

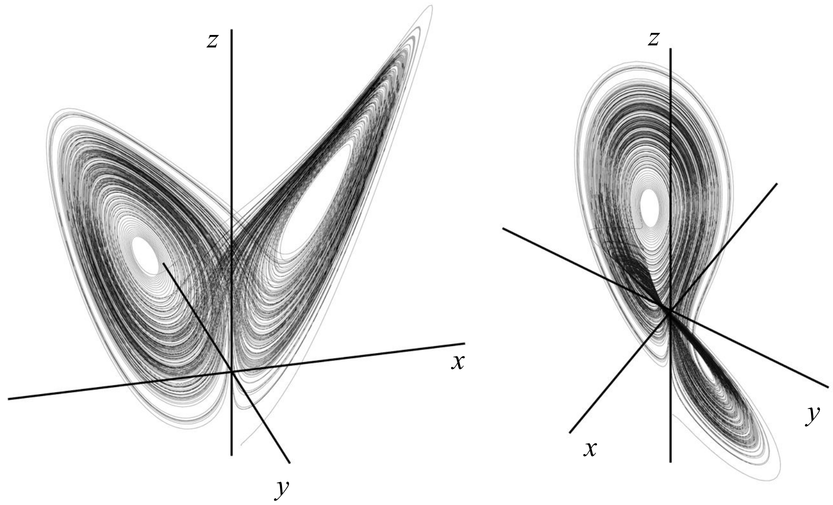

Figure 4.20. *Two views of a numerical approximation of the Lorenz Attractor for* $(\sigma, b, r) = (10, 8/3, 28)$*. The axes shown are centered at* $(0, 0, 20)$ *and are of length* 50.

The vector field on the surface of the ball can be shown to point inward if R is chosen large enough. To see this, compute the derivative of the function $C(x, y, z) = x^2+y^2+(z-r-\sigma)^2$ to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}C &= \sigma xy - \sigma x^2 + rxy - y^2 - xyz + (z - r - \sigma)(xy - bz)\\ &= -\sigma x^2 - y^2 - bz^2 + (r+\sigma)bz\\ &= -\sigma x^2 - y^2 - b\left(z - \frac{r+\sigma}{2}\right)^2 + b\frac{(r+\sigma)^2}{4}.\end{aligned}$$

Since b and σ are positive, the set on which $dC/dt = 0$ defines an ellipsoid,

$$E = \left\{(x, y, z) : \sigma x^2 + y^2 + b\left(z - \frac{r+\sigma}{2}\right)^2 = b\frac{(r+\sigma)^2}{4}\right\},$$

such that outside E, $dC/dt < 0$. To guarantee that this is true on the surface of the ball B_R for some R requires finding an R such that $B_R \subset E$. The maximum distance from the origin for points on E occurs on one of the axes; this gives the inequality

$$R > \frac{|r+\sigma|}{2}\max\left(2, \sqrt{b}, \sqrt{\frac{b}{\sigma}}\right). \tag{4.49}$$

For the classic Lorenz parameters this requirement is $R > 38$; so for example, B_{39} is a trapping region. The resulting attractor is amazingly complex, as shown in Figure 4.20.

The Lorenz attractor, Λ, is commonly visualized by numerically computing a single trajectory. Thus it appears to be the ω-limit set of an arbitrary point and qualifies as an attractor. It is not obvious, however, from the numerical simulations exactly how complicated the dynamics are on Λ: it is possible that Λ is simply a very long periodic orbit. Indeed showing that there is no attracting periodic orbit for the classic Lorenz system was listed by Stephen Smale as his 14th mathematical problem for the 21st century (Smale 1998). Recently this has been proved using rigorous numerical computation (Tucker 2002). An attractor that is geometrically complicated, such as the Lorenz attractor, is called a *strange attractor*; see §7.3. ∎

Note that not every ω-limit set is an attractor. As an example, the origin in (4.45) is the ω-limit set for any initial condition that starts on the figure eight but it is not an attractor because points in its neighborhood have limit points on the figure eight. The figure eight itself, however, is an attractor according to our definition. Note that this attractor is not a minimal set and thus does not satisfy Perko's definition of attractor.

4.11 Stability of Periodic Orbits

A periodic orbit is an invariant set and can be stable (recall example (4.43)) or unstable. It is natural to first study their stability using the same method of linearization that we used for equilibria in §4.4. Indeed, we will show that linearization provides valid results in the same situation as in that case: when the orbit is linearly asymptotically stable.

Suppose that $x(t) = \gamma(t) = \gamma(t+T)$ is a periodic orbit of period T for the differential equation $\dot{x} = f(x)$. If the vector field $f \in C^1$ we can linearize the ODE about γ by setting $x(t) = \gamma(t) + y(t)$ and expanding f in a Taylor series to obtain

$$\frac{d}{dt}(x + y) = f(\gamma(t)) + \frac{d}{dt}y = f(\gamma(t) + y) = f(\gamma(t)) + Df(\gamma(t))y + o(y).$$

If we neglect the $o(y)$ term we obtain the linearization

$$\frac{d}{dt}y = Df(\gamma(t))y = A(t)y, \tag{4.50}$$

where the matrix, $A(t)$, is a periodic function of time. Such systems can be analyzed using Floquet theory, as we did in §2.8.

Recall from (2.46) that the fundamental matrix solution of (4.50) can be written $\Phi(t, t_o)$, and that the matrix $M = \Phi(T, 0)$, is called the monodromy matrix. The eigenvalues of M are the Floquet multipliers, and Floquet's theorem (Theorem 2.13) shows that all of the solutions of (4.50) are bounded whenever the Floquet multipliers have magnitude smaller than one.

For the case (4.50), one of the Floquet multipliers is trivially unity.

Theorem 4.19. *The monodromy matrix M for the linearization of a system $\dot{x} = f(x)$ about a periodic orbit $\gamma(t)$ always has at least one unit eigenvalue.*

Proof. Since $x(t) = \gamma(t)$ is a solution of the original nonlinear equations, so is $x(t) = \gamma(t+\tau)$ for any phase shift τ. Differentiate this solution with respect to τ and set $\tau = 0$ to give

$$\frac{d}{d\tau}\left[\dot{\gamma}(t+\tau) = f(\gamma(t+\tau))\right]\Bigg|_{\tau=0} \quad \Rightarrow \quad \frac{d}{dt}\dot{\gamma} = Df(\gamma(t))\dot{\gamma}(t).$$

Therefore, $\dot{\gamma}$ is a solution of the linearized equations: $\dot{\gamma}(t) = \Phi(t, 0)\dot{\gamma}(0)$. However, since γ is periodic, $\dot{\gamma}(T) = \dot{\gamma}(0)$ and is therefore an eigenvector of the monodromy matrix with eigenvalue (Floquet multiplier) one. □

Note that the vector $\dot{\gamma}(t)$ is tangent to γ at the point $\gamma(t)$. A simple interpretation of Theorem 4.19 is that two nearby points on the same orbit stay close for all time. Since there is always a unit multiplier, a periodic orbit cannot be asymptotically stable in the same sense as an equilibrium. However, the unit multiplier is associated with the "trivial" tangent direction and does not affect the stability of the invariant set γ. Thus we will say a periodic orbit is *linearly stable* if all of its Floquet multipliers have magnitude at most 1, $|\mu_i| \le 1$. Moreover, the orbit is *linearly asymptotically stable* if all of its multipliers apart from the trivial unit multiplier have magnitude strictly less than one, $|\mu_i| < 1$ for $i = 2, \ldots, n$.

Abel's theorem, Theorem 2.11, gave one nontrivial relation between the Floquet multipliers,

$$\det(M) = \exp \int_0^T \operatorname{tr}(Df(\gamma(s)))\, ds. \tag{4.51}$$

Since $\det(M) = \prod_i \mu_i$, this relation determines the product of the multipliers. For the planar case, this is all the information we need: in $\mathbb{R}^2$, the 2×2 monodromy has one unit multiplier, $\mu_1 = 1$. The second nontrivial multiplier thus determines the stability of the periodic orbit, and $\mu_2 = \det(M)$.

Example: Consider again the planar system (4.43). Consider the limit cycle $\gamma = \{(r, \theta) = (1, \theta_o + t) : t \in \mathbb{R}\}$. Choosing $\theta_o = 0$ and returning to rectangular coordinates so that $\gamma = \{(x, y) = (\cos t, \sin t) : t \in \mathbb{R}\}$ gives the linearized matrix

$$Df(\gamma(t)) = \begin{pmatrix} -2x^2 & -1 - 2yx \\ 1 - 2yx & -2y^2 \end{pmatrix} = \begin{pmatrix} -2\cos^2 t & -1 - 2\sin t \cos t \\ 1 - 2\sin t \cos t & -2\sin^2 t \end{pmatrix}.$$

As promised, the derivative of the solution, $\dot{\gamma} = (-\sin t, \cos t)^T$, is a solution of the linearized ODE:

$$\frac{d}{dt}\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} = \begin{pmatrix} -2\cos^2 t & -1 - 2\sin t \cos t \\ 1 - 2\sin t \cos t & -2\sin^2 t \end{pmatrix}\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} = \begin{pmatrix} -\cos t \\ -\sin t \end{pmatrix}.$$

A second solution can be easily obtained by linearizing the r equation (4.44) about its equilibrium $r = 1$, to obtain $\delta\dot{r} = -2\delta r$, showing that a linearized solution should take the form $(\delta x, \delta y) = \delta r_o e^{-2t}(\cos t, \sin t)$. Indeed, substituting this into the linearized ODE yields an identity. We can conclude that the fundamental matrix solution to the linear equation is

$$\Phi(t, 0) = \begin{pmatrix} e^{-2t}\cos t & -\sin t \\ e^{-2t}\sin t & \cos t \end{pmatrix},$$

which gives a monodromy matrix

$$M = \Phi(2\pi, 0) = \begin{pmatrix} e^{-4\pi} & 0 \\ 0 & 1 \end{pmatrix}.$$

The Floquet multipliers are simply the elements on the diagonal, $\mu_1 = 1$ and $\mu_2 = e^{-4\pi}$. ■

If $\mathrm{tr}(Df)$ vanishes identically, then (4.51) implies that $\det(M) = 1$; this means that a planar, "incompressible" flow has both multipliers equal to one (see §9.2).

Example: Any C^2 Hamiltonian system in the plane, (4.27), has both Floquet multipliers equal to one, since $f = (\partial H/\partial y, \partial H/\partial x)$, so that $\mathrm{tr}(Df) = \partial^2 H/\partial x \partial y - \partial^2 H/\partial y \partial x = 0$. If one is careful with indices, one can show that $\mathrm{tr}(Df) = 0$ for Hamiltonian systems in any dimension (recall (1.13)), which means that the product of the Floquet multipliers for these systems is always one.

If the Hamiltonian depends explicitly on time, $H(x, y, t)$, the system (4.27) is still called a Hamiltonian system; however, the energy is no longer conserved. Indeed, (4.28) becomes

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t} \neq 0.$$

As we discussed in §1.2, a two-dimensional nonautonomous system is equivalent to an autonomous one on a three-dimensional space. If we suppose that H is a periodic function of time, $H(x, y, t) = H(x, y, t + T)$, then the third variable can be taken to be an angle, say, $\theta = t/T$, so the phase space is $M = \mathbb{R}^2 \times \mathbb{S}^1$, and the ODEs are

$$\dot{x} = \frac{\partial}{\partial y} H(x, y, T\theta), \quad \dot{y} = -\frac{\partial}{\partial x} H(x, y, T\theta), \quad \dot{\theta} = \frac{1}{T}.$$

A periodic orbit of this system is a curve $\gamma(t) = (x(t), y(t), \theta(t))$ whose period must be some multiple of T, since the angle returns to itself "mod 1." Since the third component of the new three-dimensional vector field is constant, the result $\mathrm{tr}(Df) = 0$ still holds. In this case there are three Floquet multipliers. One multiplier will be one, $\mu_1 = 1$, and so $\mu_2\mu_3 = 1$ as well. ∎

Consequently, periodic orbits of Hamiltonian systems are never asymptotically stable. The only case in which they are linearly stable is if all Floquet multipliers are on the unit circle. This will be discussed in Chapter 9.

The relationship between linear asymptotic stability and true asymptotic stability in the sense of §4.10 is most easily discussed by introducing the concept of Poincaré maps.

4.12 Poincaré Maps

Maps are dynamical systems in the sense of §4.1 when the set of allowed time values is discrete. While much of the theory of dynamical systems can be developed for maps themselves (Arrowsmith and Place 1992; Devaney 1986; Guckenheimer et al. 1983; Katok and Hasselblatt 1999; Robinson 1999; Strogatz 1994; Wiggins 2003), our primary interest in maps will be to discuss the behavior of a flow in the neighborhood of a periodic orbit. The Poincaré map naturally arises in this context.

A map is defined by a function $P : M \to M$ through the relation $x' = P(x)$, where $x' \in M$ denotes the new point that arises from the initial point $x \in M$.[29] For a map, an

[29] We always use the symbol "D" to represent derivative and reserve the prime symbol $'$ for the iterate of a map.

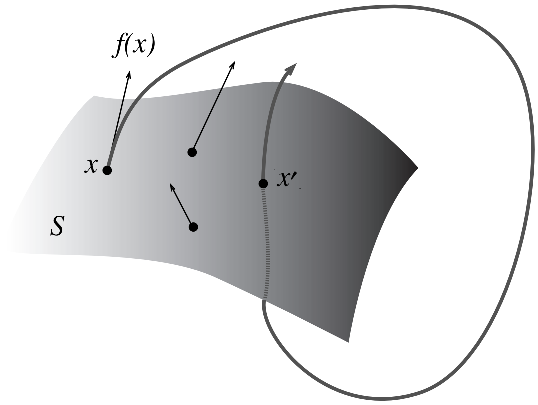

Figure 4.21. *Construction of a Poincaré map from a flow on a section S.*

orbit is no longer a function $x(t)$ of $t \in \mathbb{R}$ but is instead a sequence $\{x_t : t \in \mathbb{Z}\}$. Using this subscript notation, the dynamics is given by the iteration

$$x_t = P(x_{t-1}).$$

Maps arise naturally from flows by taking *sections* of the flow. For a flow in $\mathbb{R}^n$, a section, S, is a surface of dimension $d = n - 1$ (i.e., a codimension-one surface) such that the velocity vector is not tangent to S at any point. That is, if $\hat{n}_x$ is the unit normal to S at x, then S is a section if $f(x) \cdot \hat{n}_x \neq 0$ for all $x \in S$.

A *Poincaré map* for a section S is obtained by choosing an $x \in S$, and following the flow $\varphi_t(x)$ to find the first return to S: let $\tau(x)$ be the first positive time for which $\varphi_t(x) \in S$. The map is defined by

$$P(x) = \varphi_{\tau(x)}(x), \tag{4.52}$$

as illustrated in Figure 4.21. Note that $\tau(x)$ might not exist for all $x \in S$, in which case the Poincaré map is not well defined. The best scenario occurs when S is a

> ▷ *global section*: If the orbit of every point $x \in \mathbb{R}^n$ eventually crosses an $n-1$ dimensional surface S and then returns to S at a later time, then S is a global section.

In this case the Poincaré map is defined for all $x \in S$.

Example: A system with a natural angle variable that is always increasing has a global section. For example, the skew-product[30] system

$$\dot{x} = f(x, \theta),$$
$$\dot{\theta} = 1,$$

[30]A system $\dot{x} = f(x)$ is a skew product if the variables can be separated as $x = (y, z)$ such that the equations become $\dot{y} = f_1(y, z)$ and $\dot{z} = f_2(z)$.

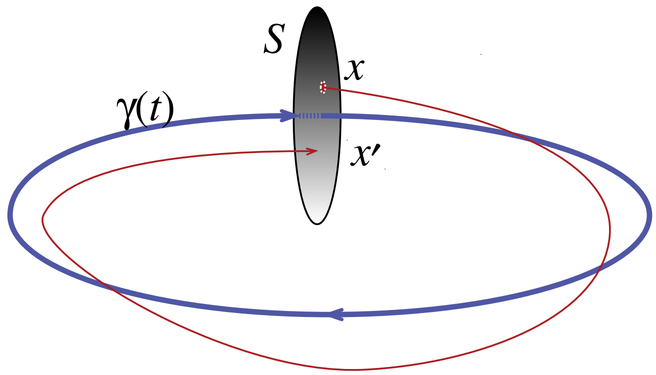

Figure 4.22. *Poincaré section in the neighborhood of a periodic orbit.*

where $x \in \mathbb{R}^n$, and $\theta \in \mathbb{S}^1$, has a global section $S = \{(x, \theta) : \theta = \theta_o\} \cong \mathbb{R}^{n-1}$, since all trajectories cross this section with unit speed in the θ direction. This can also be generalized to the case that $\dot{\theta} = g(x, \theta)$, provided that $g > 0$ everywhere. ∎

If S and $\tilde{S}$ are two global sections, then the corresponding Poincaré maps are conjugate. This follows since the flow takes every point $x \in S$ to a point on $\tilde{S}$ after some time $\tau(x)$. The homeomorphism $h : S \to \tilde{S}$ is defined by $h(x) = \varphi_{\tau(x)}(x)$. Each global section contains the same information about the flow.

A locally defined Poincaré map always exists in a neighborhood of a periodic orbit γ, as shown in Figure 4.22. The section S is assumed to be a (small) disk containing a point $x_o \in \gamma$ that is oriented perpendicular to the vector field $f(x_o)$. By continuity, there is always some neighborhood of this point for which the vector field will be transverse to the disk. Moreover, continuity with respect to initial conditions, recall §3.4, implies that points "near" γ will stay "near" for any finite time t, and so they must intersect the disk at a time that is near the period, $T = \tau(x_o)$.

For example, suppose that a flow in the plane has a periodic orbit. Then the section is a line segment that is perpendicular to the periodic orbit at a point on the orbit.

Example: Let (r, θ) be polar coordinates and consider the system

$$\begin{aligned} \dot{r} &= r + \alpha r^3, \\ \dot{\theta} &= \nu. \end{aligned}$$

When $\alpha < 0$ there is a unique periodic orbit at $r^* = (-\alpha)^{-1/2}$. It is not hard to solve explicitly for $r(t)$ by separation of variables:

$$t + c = \int \frac{dr}{r(1+\alpha r^2)} = \frac{1}{2} \ln \left| \frac{r^2}{1+\alpha r^2} \right| \quad \Rightarrow \quad r(t; r_o) = \frac{r_o}{\sqrt{(1+\alpha r_o^2)e^{-2t} - \alpha r_o^2}}.$$

The solution for θ is trivial: $\theta(t) = \theta_o + \nu t$. Let the positive x-axis represent S. The radius r is a good coordinate on S and the Poincaré map $P : S \to S$ is simply the value that r

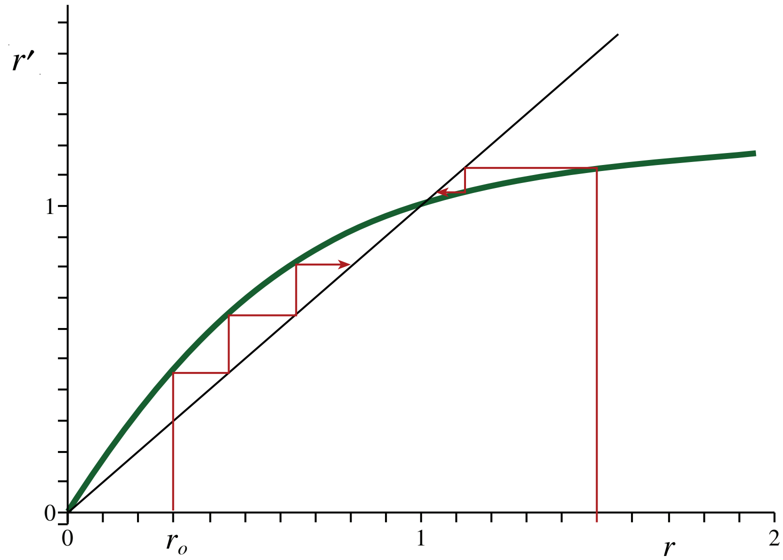

Figure 4.23. *Poincaré map* (4.53) *for* $\alpha = -1$ *and* $\nu = r\pi$*. The periodic orbit corresponds to the intersection of the graph* $r' = P(r)$*. It is stable because* $DP(1) < 1$*. The stair-stepped curve is the graphical iteration of* $r_o = 0.3$.

takes after one period of the angle, or at $t = 2\pi/\nu$:

$$r' = P(r) = \frac{r}{\sqrt{(1+\alpha r^2)e^{-4\pi/\nu} - \alpha r^2}}. \tag{4.53}$$

For this one-dimensional case, the Poincaré map and its iteration can be visualized graphically; see Figure 4.23. Consider an initial condition r_o. Move vertically up to $P(r_o)$ to obtain r_1. Put this value onto the r-axis by moving horizontally to the diagonal. To get r_2 move again vertically to the function value $P(r_2)$. The resulting series of lines, as shown in the figure, resembles a staircase. (For more complicated maps the picture looks like a cobweb and so is typically called the *cobweb diagram*.) The staircase picture implies that if the slope at a fixed point is less than one in magnitude, then the equilibrium is stable, since iterates move monotonically in the direction of the fixed point. ∎

Generally, the computation of the stability of a periodic orbit requires that we consider the linearization of the flow in the neighborhood of the periodic orbit. One must typically resort to numerical methods to solve for the Floquet multipliers, even if the periodic orbit is known analytically. It is often convenient numerically to compute the Poincaré map (4.52) and study stability of an orbit by this method. One advantage is that the Poincaré map acts on the section S that has dimension $n-1$, one less than the flow. Moreover, the removed dimension corresponds to the motion along the periodic orbit and thus to the neutral Floquet multiplier $\mu_1 = 1$. Consequently, stability computed using the Poincaré map is the same as that from the Floquet spectrum:

Theorem 4.20. *Let γ be a periodic orbit of a C^2 flow φ, S be a local section through a point $x_o \in \gamma$, and $P : S \to S$ be the Poincaré return map. If the monodromy matrix of γ is M, then*

$$\operatorname{spec}(M) = \operatorname{spec}(DP(x_o)) \cup \{1\}.$$

Proof. Suppose $x \in S$, and $\tau(x)$ is the time of first return to S. The Poincaré map is given by (4.52), where we restrict x to S. For the moment, ignore this restriction, and let $Q(x) = \varphi_{\tau(x)}(x)$ for any x near γ. Differentiating Q with respect to x gives

$$DQ(x) = D_x\varphi_{\tau(x)}(x) + \frac{d}{dt}\varphi_{\tau(x)}(x)\,(D_x\tau(x))^T\,.$$

Here the last term is the "outer product" of the flow vector $f(x(\tau(x))$ and the gradient vector $D\tau(x)$. This latter vector represents the change in period with respect to x; it can be called the "twist." When $x = x_o \in \gamma$, $\tau(x_o) = T$, $D\varphi_T(x_o) = M$, and $\varphi_T(x_o) = x_o$ so that

$$DQ(x_o) = M + f(x_o)(D\tau(x_o))^T.$$

We can take the section S to consist of points orthogonal to the flow vector at x_o, i.e., $x = x_o + \xi$, where $f(x_o)^T\xi = 0$. If w_i, $i = 1, 2, \ldots, n-1$, are a set of orthonormal basis vectors perpendicular to $f(x_o)$, then the transpose of the $n \times (n-1)$ matrix $W = (w_1, w_2, \ldots, w_{n-1})$ is a projection onto vectors in the section. The matrix $DP(x_o)$ in the w_i basis has the representation $W^T DQ(x_o)W$. Since $W^T f(x_o) = 0$, we obtain

$$DP(x_o) = W^T MW.$$

Consequently, if v is an eigenvector of $DP(x_o)$ with eigenvalue μ, then since $WW^T = I$, the $(n-1)\times(n-1)$ identity matrix, Wv is an eigenvector of M with the same eigenvalue. The only vector not in the projected space is $f(x_o)$, which is an eigenvector of M with eigenvalue one. $\square$

This theorem shows that, up to the trivial Floquet multiplier, $\mu_1 = 1$, linear stability of a periodic orbit can be computed from the Poincaré map.

Finally we are ready to state the result about linear stability.

Theorem 4.21. *If γ is a periodic orbit of a C^2 flow that is linearly asymptotically stable (the spectrum of its Poincaré map is inside the unit circle), then it is asymptotically stable.*

Proof. The proof of this theorem is similar to the proof of Theorem 4.6. Following that analysis, let $x_o \in \gamma$, and $x_o + y \in N \cap S$, where N is a neighborhood of γ and S is a section. Write the Poincaré map at $x_o + y$ as $P(x_o + y) = x_o + DP(x_o)y + g(y)$. Thus

$$y' = DP(x_o)y + g(y).$$

Since the orbit γ is linearly asymptotically stable, the spectrum of $DP(x_o)$ is contained in the interior of the unit circle. Analogously to (4.20), for any $n \geq 0$ we can bound the orbit of this linear mapping by

$$\left|DP^n(x_o)y\right| < K\mu^n\,|y|$$

for some $0 < \mu < 1$ and $K \geq 1$. Since the flow is smooth, $g(y) = o(y)$, that is, for any ε there is a neighborhood $N_\varepsilon \subset S$ of x_o such that $|g(y)| < \varepsilon\,|y|$ for all $y \in N_\varepsilon$. Using the discrete analogue of the integrating factor and the Grönwall lemma, it is possible to see that there is an ε such that if $y_o \in N_\varepsilon$, then the sequence y_n limits to x_o as $n \to \infty$ and is bounded in distance from x_o. We leave the details to the reader. Since the Poincaré maps through any two local sections to γ are topologically conjugate, this implies that γ is asymptotically stable. $\square$

4.13 Exercises

1. Show that the following functions are flows on the spaces indicated. Find the vector field for each flow.

   (a) $\varphi_t(x) = \dfrac{x + \tanh t}{1 + x \tanh t}$, $x \in [-1, 1]$,

   (b) $\varphi_t(x, y) = \begin{pmatrix} x\cos(r^2 t) + y\sin(r^2 t) \\ -x\sin(r^2 t) + y\cos(r^2 t) \end{pmatrix}$, $\;r^2 = x^2 + y^2$, $\;(x, y) \in \mathbb{R}^2$.

2. Find and analyze the linear behavior near each equilibrium of the following systems on $\mathbb{R}^2$. Classify the equilibria. Are they linearly stable or unstable? Sketch the local behavior you obtained in the phase plane and compare with a numerical phase plane plotter that shows the global solutions.

   (a) $\begin{array}{l} \dot{x} = y \\ \dot{y} = x - x^3 - ay \end{array}$,

   (b) $\begin{array}{l} \dot{x} = x^2 - y^2 - 1 \\ \dot{y} = 2y \end{array}$,

   (c) $\begin{array}{l} \dot{x} = y - x^2 + 2 \\ \dot{y} = 2y^2 - 2xy \end{array}$,

   (d) $\begin{array}{l} \dot{x} = -4x - 2y + 4 \\ \dot{y} = xy \end{array}$.

3. The centrifugal governor (see Figure 4.24) was patented by James Watt in 1789 to control the steam engine. It is described by the set of ODEs (Pontryagin 1962)

   $$\begin{aligned} \dot{\varphi} &= \psi, \\ \dot{\psi} &= n^2\omega^2 \sin\varphi\cos\varphi - \Omega^2\sin\varphi - \tfrac{b}{m}\psi, \\ \dot{\omega} &= \tfrac{1}{I}\left(\mu\cos\varphi - F\right), \end{aligned}$$

   similar to those first derived by Vishnegradskii in 1876. Here the dynamical variables are $\varphi \in [0, \pi]$, the angle between the spindle S and the "flyball arms" of length L, ω, the rotational velocity of the flywheel, and ψ, the angular acceleration. Constants in the equation are n the transmission ratio of the gears—the ratio between the angular velocity of the spindle and flywheel, $\Omega = \sqrt{g/L}$ the arm pendulum frequency, b friction of the flywheel, m the flyball mass, I the moment of inertia of the flywheel, F the torque load on the engine, and μ, representing the steam-driven torque caused by closing the valve as the collar rises on the spindle.

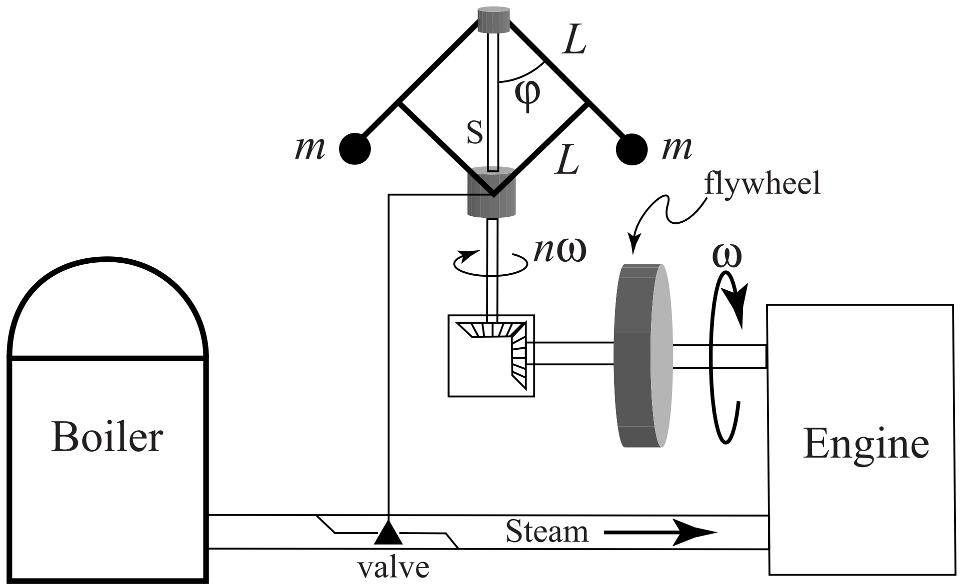

Figure 4.24. *Sketch of Watt's centrifugal governor.*

(a) Show that by rescaling time, setting $\tau = \Omega t$, and defining new variables, $(x, y, z) = \left(\varphi, \psi/\Omega, n\omega/\Omega\right)$, the equations can be reduced to the system

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= \sin x\left(z^2\cos x - 1\right) - \varepsilon y,\\ \dot{z} &= \alpha\left(\cos x - \beta\right)\end{aligned}$$

for new parameters $(\alpha, \beta, \varepsilon)$, all positive.

(b) Show that if β is small enough, there is a unique the equilibrium (x^*, y^*, z^*).

(c) Linearize about the equilibrium and find the characteristic polynomial.

(d) Show that there is a critical value, $\varepsilon_o(\alpha, \beta)$, such that if $\varepsilon > \varepsilon_o$, then the equilibrium is asymptotically stable, and if $0 < \varepsilon < \varepsilon_o$, then the equilibrium is a saddle.

(e) It can be shown that the system undergoes a Hopf bifurcation (see Chapter 8) at ε_o. Solve the equations numerically and demonstrate that as ε decreases through ε_o the equilibrium becomes unstable and there is an attracting limit cycle.

4. Are the following functions homeomorphisms? Are they diffeomorphisms? If the functions depend upon parameters, then so might your answers. Explain.

(a) $f : [0,1] \to [0,1], \quad f(x) = ax(1-x)$,

(b) $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = ax + b\sin(2\pi x)$,

(c) $f : [0,1] \to \mathbb{S}^1, \quad f(x) = [x + b\sin(2\pi x)] \bmod 1$,

(d) $f : \mathbb{S}^1 \times \mathbb{R} \to \mathbb{S}^1 \times \mathrm{R}, \quad f(x, y) = ([x+y+b\sin(2\pi x)] \bmod 1, y + b\sin(2\pi x))$,

(e) $f : \mathbb{R}^2 \to \mathbb{R}^2, \quad f(x, y) = (y + ax(1 - x), -bx)$.

5. Use the iterative construction of the Hartman–Grobman homeomorphism H to obtain an approximation for the conjugacy for the flow of the system on $\mathbb{R}^3$ given by

$$\begin{aligned}\dot{x} &= -x,\\ \dot{y} &= -y + x^2 z,\\ \dot{z} &= 2z\end{aligned}$$

to its linearization at $(0, 0, 0)$. Show that the iteration is not globally convergent. Discuss how to modify the iteration to make it locally convergent, using a "bump function."

6. Which of the ODEs $\dot{x} = Ax$ with the following matrices are topologically conjugate? Which are diffeomorphic? Which are linearly conjugate?

(a) $\begin{pmatrix} -2 & -1 \\ 3 & 2 \end{pmatrix}$, (b) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, (c) $\begin{pmatrix} -5 & -2 \\ 5 & 1 \end{pmatrix}$, (d) $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$,

(e) $\begin{pmatrix} 7 & -10 \\ 5 & -8 \end{pmatrix}$, (f) $\begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$, (g) $\begin{pmatrix} -5 & 1 \\ -6 & 0 \end{pmatrix}$, (h) $\begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix}$.

7. Construct a topological conjugacy between the linear systems with the matrices

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

(*Hint*: Transform to polar coordinates and assume the homeomorphism has the form $h(r, \theta) = (h_r(r), h_\theta(r, \theta))$. The r-dependence of h_θ will involve $\ln r$.)

8. Construct Lyapunov functions to determine the stability of the equilibrium $(0, 0)$ for the following systems on $\mathbb{R}^2$.

(a) $\begin{aligned}\dot{x} &= -x + y - y^2 - x^3\\ \dot{y} &= x - y + xy\end{aligned}$,

(b) $\begin{aligned}\dot{x} &= y - x^2 + 3y^2 - 2xy\\ \dot{y} &= -x - 3x^2 + y^2 + 2xy\end{aligned}$.

(*Hints*: Try a power series for L, starting with quadratic terms. Add higher-order terms if necessary. Sometimes it is easier to check for a Hamiltonian than it is to construct L *ab initio*.)

9. An asymptotically stable linear system always has a Lyapunov function of the form $L = x^T S x$.

   (a) Show that when all the eigenvalues of A have negative real parts, then the "Lyapunov equation" (4.24) has the unique, positive definite, symmetric solution

$$S = \int_0^\infty e^{\tau A^T} e^{\tau A} d\tau. \tag{4.54}$$

(*Hint*: Premultiply (4.24) by e^{tA^T} and postmultiply by e^{tA}. Note that the left-hand side of (4.24) then becomes a total derivative. Remember that $e^{A^T+A} \neq e^{A^T}e^{A}$ in general.)

(b) Compute S for the matrix $A = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}$, and demonstrate explicitly that $dL\big/dt < 0$.

10. The Lyapunov function defined in Exercise 9 also works when nonlinear terms are added to the ODE. Consider the system $\dot{x} = Ax + g(x)$, where $g(x) = o(x)$ and A is a matrix whose eigenvalues have negative real parts. Show that there is a neighborhood U of the origin for which the function $L = x^T Sx$, where S is given by (4.54), is a strong Lyapunov function. (*Hint*: You may need to use the Cauchy–Schwarz inequality $|\langle u, v\rangle| \leq \|u\| \; \|v\|$.)

11. In 1965 Goodwin proposed the model

$$\dot{x} = \frac{1}{1+z^m} - ax, \quad \dot{y} = x - by, \quad \dot{z} = y - cz,$$

for the regulation of enzyme synthesis of a product in a cell. Here a, b, c are positive constants, and m is a positive integer ($m = 1$ for Goodwin's original model) (Murray 1993, §6.2). Here x represents the concentration of messenger RNA, y the enzyme, and z the product. The nonlinear term in these equations represents the negative feedback of the product on the RNA, since as z grows, the growth rate of x decreases.

(a) Show that there is a trapping set of the form $N = \{(x, y, z) : 0 \leq x \leq X, 0 \leq y \leq Y, 0 \leq z < Z\}$ for suitably chosen values X, Y, and Z. Take care to think about the dynamics on the coordinate axes.

(b) Find the unique equilibrium in N, and show that it is asymptotically stable when $m = 1$. It also can be shown with more work that this is true for any $m < 8$. (*Hint*: The characteristic polynomial has only stable roots only if it satisfies the Routh-Hurwitz criterion; see Exercise 2.11.) Consequently the attracting set in Λ contains this equilibrium. While this system was initially proposed to model oscillatory behavior, a recent general result implies that no such cycle exists for $m \leq 4$ and indeed that the attracting set Λ in N is the equilibrium (Enciso and Sontag 2006).

12. Assume that the flow $\varphi_t : A \to A$ is conjugate to the flow $\psi_t : B \to B$ with conjugacy $h : A \to B$.

(a) Show that if $\omega(x)$ is the omega limit set for $x \in A$ under φ, then $h(\omega(h^{-1}(y)))$ is the omega limit set for $y = h(x) \in B$ under ψ.

(b) Show that if Λ is an invariant set for φ, then $h(\Lambda)$ is an invariant set for ψ.

(c) Show that if $W^s(\Lambda)$ is the basin of Λ, then $h(W^s(\Lambda))$ is the basin of $h(\Lambda)$.

(d) Show that if Λ is an attractor, then so is $h(\Lambda)$.

13. Suppose that φ and ψ are flows on $\mathbb{R}^2$ that each have exactly two equilibria that are both saddles. Suppose for the flow φ that the unstable set of one saddle corresponds to the stable set of the other but that this is not true for ψ. Show that φ and ψ are not topologically equivalent.

14. Show that if $y \in \omega(x)$, then y is nonwandering.

15. An alternative trapping set to (4.48) for the Lorenz system (4.26) is the ellipsoid

$$E_C = \left\{ rx^2 + \sigma y^2 + \sigma(z - 2r)^2 \le C \right\}.$$

Find the minimal value of C such that every trajectory eventually enters E_C. Does this give a better bound than that represented by (4.49)?

16. Let (r, θ) be a point in the phase space $\mathbb{R}^+ \times \mathbb{S}$ that obeys the system

$$\begin{aligned} \dot{r} &= r(1 + a\cos\theta - r^2), \\ \dot{\theta} &= 1, \end{aligned}$$

where $|a| < 1$.

(a) Show that the circle $r = 0$ is periodic orbit with period 2π.

(b) Compute the monodromy matrix $M = \Phi(2\pi, 0)$ for the circle $r = 0$ and show that its Floquet multipliers are $\mu = 1$ and $e^{2\pi}$. (*Hint*: The linear system has solutions $(0, \delta\theta(t))$ and $(\delta r(t), 0.)$

(c) Show that there are two circles $r = r_-$ and $r = r_+$ such that if $0 < r < r_-$, then $\dot{r} > 0$, and if $r > r_+$, then $\dot{r} < 0$. Thus the region $N = \{(r, \theta), r_- < r < r_+\}$ is a trapping region. Our next goal is to show that the attracting set in N is a periodic orbit.

(d) Let S be the ray $\{(r, 0)\}$. Argue that S is a global section. Let $P : \mathbb{R}^+ \to \mathbb{R}^+$ be the Poincaré map on S.

(e) Suppose that the orbit of the point $(r_L, 0)$ has the property $0 < P(r_L) < r_-$. Argue that $P(r_L) > r_L$. Alternatively, suppose that the orbit of$(r_H, 0)$ has the property that $P(r_H) > r_+$. Then argue that $P(r_H) < r_H$.

(f) Apply the intermediate value theorem to $P(r)$ to show that there is a point $(r^*, 0)$, where $r_L < r^* < r_H$, whose orbit is periodic.

(g) Show that the Floquet multipliers of the new orbit are $\mu = 1$ and $e^{-4\pi}$. Consequently, the new periodic orbit is asymptotically stable. (*Hint*: To do the integral $\int_0^{2\pi} r^2(t)dt$ use the differential equation to set $r^2 = 1 + a\cos\theta - \dot{r}/r$.)

17. The Shimizu–Morioka model is a simplified model of the Lorenz system when r is large (Shilnikov 1993). It is given by

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x - \alpha y - xz, \\ \dot{z} &= -\beta z + x^2, \end{aligned}$$

where $(x, y, z) \in \mathbb{R}^3$, and $\alpha, \beta \in \mathbb{R}$.

(a) Find all of the equilibria for this system depending the values of α and β (there can be three).

(b) Find the eigenvalues of the equilibrium that exists (is a point in $\mathbb{R}^3$) for all parameter values, and classify its stability type as a function of α and β.

18. Consider your adopted system of quadratic differential equations (recall §1.6 and Exercise 1.10). If possible, find a set of values of the reduced parameters for which one of your systems equilibria (x^*, y^*, z^*) is spectrally stable. If there are no such equilibria, then prove so. Otherwise, attempt to construct a Lyapunov function for a neighborhood of your stable equilibrium. It would probably be good to attempt to use a quadratic function

$$L(x, y, z) = \alpha(x - x^*)^2 + \beta(y - y^*)^2 + \gamma(z - z^*)^2,$$

though you might have to experiment with adding cross terms to the equation, or going to a higher degree. This is a case where you may or may not succeed; indeed, your system may not have a simple Lyapunov function. You will get full credit for making a convincing attempt—for example, by showing that the function above is not a Lyapunov function for any values of α, β, γ.

Chapter 5

Invariant Manifolds

Nunquam praescriptos transibunt sidera fines. (Never will heavenly bodies transgress their prescribed bounds.) (Henri Poincaré 1890)

Hyperbolic fixed points of a linear ordinary differential equation (ODE) have stable, E^s, and unstable spaces, E^u, determined by the eigenvectors of the associated matrix at the fixed point. We showed in §2.6 that these spaces are invariant under the dynamics of the linear system. In this chapter we will show that there are also invariant subspaces W^u and W^s that are generalizations of E^u and E^s for a nonlinear ODE with a hyperbolic fixed point. Some local information about these subspaces can be inferred from Theorem 4.12 (Hartman–Grobman), which implies that when an equilibrium is hyperbolic, the flow in its neighborhood is topologically conjugate to the linearized flow. Here, however, we will obtain much more precise control over the structure of these subspaces, showing that they are "manifolds" that are smoothly tangent to the linear subspaces. We begin by looking at a few simple examples where the manifolds can be found analytically.

5.1 Stable and Unstable Sets

Stable and unstable sets are collections of orbits that are *forwar*d or *backward asymptotic* to a given orbit. Recall that in §4.10 we defined the stable set, or basin of attraction, of an invariant set Λ as the set of points forward asymptotic to Λ:

$$W^s(\Lambda) = \{x \notin \Lambda : \varphi_t(x) \to \Lambda \text{ as } t \to \infty\} . \tag{5.1}$$

We can also define the backward basin or unstable set of Λ as the set of points that are backward asymptotic to it:

$$W^u(\Lambda) = \{x \notin \Lambda : \varphi_t(x) \to \Lambda \text{ as } t \to \infty\} . \tag{5.2}$$

Generally the stable and unstable sets are invariant.

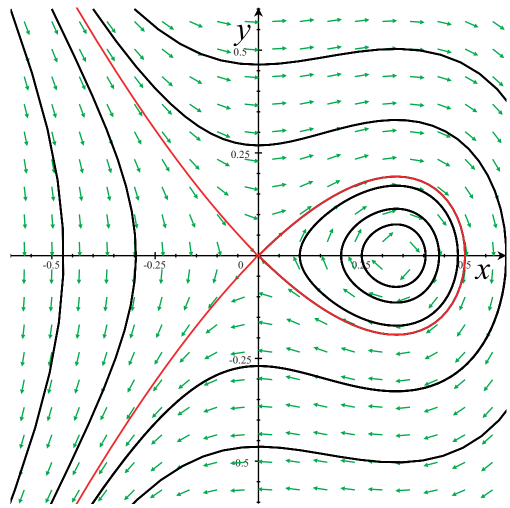

Figure 5.1. *Phase portrait of* (5.3) *with* $a = 1$.

Lemma 5.1. *The stable and unstable sets of an invariant set* Λ *are themselves invariant sets.*

Proof. We must show that whenever $z \in W^s(\Lambda)$ we have $\varphi_s(z) \in W^s(\Lambda)$ for any $s \in \mathbb{R}$. This follows from the group property of the flow: by definition (5.1), $\varphi_s(z)$ is a point such that $\varphi_t(\varphi_s(z)) = \varphi_{s+t}(z) \to \Lambda$ as $t \to \infty$. Since this holds for any s, the stable set is invariant. A similar argument applies to the unstable set. $\square$

In some special cases we can find the stable and unstable sets analytically. For example, consider a Hamiltonian $H(x, y)$ in the plane with a saddle equilibrium at a point (x^*, y^*). The energy contours $H(x, y) = H(x^*, y^*) = E$ that emanate from the saddle correspond to the stable and unstable sets of the saddle—since these are curves they are called the stable and unstable *manifolds*.

Example: The Hamiltonian for the system (4.29) is

$$H(x, y) = \frac{1}{2}(y^2 - x^2) + ax^3, \tag{5.3}$$

where we take $a > 0$. Since the linearization for the equilibrium at the origin has the Jacobian $Df(0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, it is a saddle. The energy at the saddle point is $H(0, 0) = E = 0$; this contour corresponds to the curves $y_{\pm} = \pm x\sqrt{1 - 2ax}$, shown in Figure 5.1, that intersect at $x = (2a)^{-1}$. Since orbits lie on contours of constant H, the union of these two curves, like every contour of H, is an invariant set. Noting the direction of the flow (from $\dot{x} = y$), we see that

$$\begin{aligned} W^u(0,0) &= \{(x, y) : H(x, y) = 0,\ \ x > 0 \text{ or } x, y < 0\}, \\ W^s(0,0) &= \{(x, y) : H(x, y) = 0,\ \ x > 0 \text{ or } x < 0 \text{ and } y > 0\}. \end{aligned}$$

Here we specifically do not include the equilibrium as part of the stable and unstable sets. Note that the positive-x branches of the two manifolds coincide; moreover, these branches bound the set of orbits that are oscillating about the center equilibrium at $((3a)^{-1}, 0)$. Orbits outside this closed loop are unbounded. Since this loop separates two topologically distinct types of motion, we call it a "separatrix"; see §5.2. ∎

When the ODE is linear and hyperbolic, $\mathbb{R}^n = E^s \oplus E^u$ and the stable and unstable sets of the origin correspond to E^s and E^u. Our task in this chapter is to generalize these subspaces to the nonlinear case. We will see that when the equilibrium is hyperbolic, its linear stable and unstable sets give a "linear approximation" to the stable and unstable manifolds of the equilibrium.

Example: For the Hamiltonian (5.3), the stable and unstable manifolds of the origin correspond to the curves $y_\pm = \pm x\sqrt{1-2ax}$; recall Figure 5.1. As we will see in §5.4, the stable manifold theorem implies that the local unstable manifold is the unique invariant curve emanating from the origin that is tangent to the unstable eigenvector of $Df(0)$, in this case the vector $v_+ = (1, 1)^T$. Since $dy_+ \big/ dx = 1$ at $x = 0$, this shows that the local unstable manifold of the origin is indeed the set $W^u(0) = \{(x, y_+(x)) : x \in (-\infty, 1\big/2a)\}$. Similarly, the local stable manifold is $W^s(0) = \{(x, y_-(x)) : x \in (-\infty, 1\big/2a)\}$ and is tangent to the stable eigenvector $v_- = (1, -1)^T$. ∎

5.2 Heteroclinic Orbits

In special situations it is possible that $W^u(\Lambda)$ and $W^s(\Lambda)$ may coincide or perhaps have points of intersection. The realization that there could be such intersections (in particular transverse intersections) is what led Poincaré to understand that the dynamics of the n-body problem (n point masses interacting under their mutual gravitational attraction) could be very complicated. The discovery of this complexity—and indeed the beginnings of what we now call *chaos*—arose from a mistake in a manuscript that Poincaré had submitted in 1888 to King Oscar of Sweden for a mathematics prize to be awarded to the first person to "find a solution" to the n-body problem! Although Poincaré was awarded the prize in 1889, his initial essay had mistakenly asserted that if W^u intersects W^s, then they must coincide.[31] The story of this mistake and its subsequent correction (leading to Poincaré having to pay for the entire print run of the issue of *Acta Mathematica* containing the original essay) is elegantly told in (Diacu and Holmes 1996).

The corrected version of Poincaré's paper (Poincaré 1889) began his extensive study of the complexity induced by two types of orbits; the first type he calls a

> ▷ *heteroclinic orbit*: An orbit Γ is heteroclinic if each $x \in \Gamma$ is backward asymptotic to an invariant set A and forward asymptotic to an invariant set B, i.e., $\Gamma \subset W^u(A) \cap W^s(B)$.

The second class is a special case of the first; Poincaré called the second type a *doubly asymptotic* or

> ▷ *homoclinic orbit*: Γ is homoclinic if each $x \in \Gamma$ is both forward and backward asymptotic to the same invariant set A, i.e., $\Gamma \subset W^u(A) \cap W^s(A)$.

[31] Some of the consequences of noncoincident intersections are discussed in §8.13 et seq.

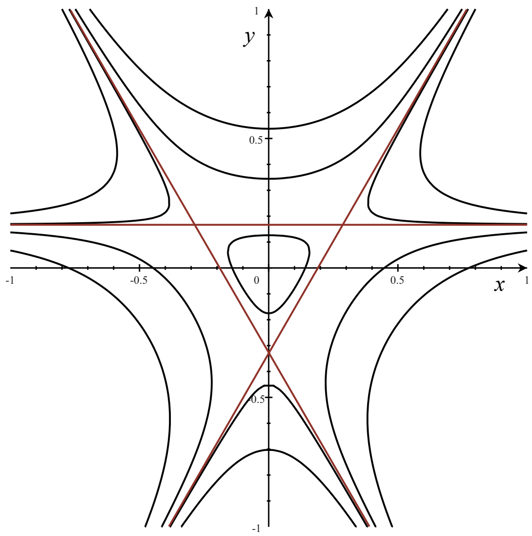

Figure 5.2. *Contours of the Hamiltonian* (5.4).

This definition could be generalized to say that an orbit Γ_h is homoclinic to another orbit Γ if every point on Γ_h is both forward and backward asymptotic to Γ.

In a two-dimensional phase space, a saddle equilibrium has both a stable and an unstable set and each is one-dimensional. The uniqueness theorem implies that if a branch of W^u intersects a branch of W^s, then they must coincide; therefore, in a two-dimensional phase space homoclinic orbits form impenetrable boundaries—we saw such a boundary in Figure 5.1. Orbits such as these are called *separatrices*, as they separate phase space into regions that cannot communicate. Poincaré's mistake in 1888 was the conclusion that this must happen in higher-dimensional systems; we will see how this fails in §8.13.

For the case of Hamiltonian systems in the plane, separatrices are common. Since H is constant along trajectories, recall (4.28), any closed contour of a Hamiltonian H that intersects one or more critical points (note that $\nabla H = 0$ implies also that the point is an equilibrium) gives a separatrix. When a heteroclinic orbit connects two saddle equilibria, it is also called a *saddle connection*.

Example: Heteroclinic orbits can be constructed by choosing an H that has several saddle points with the same energy. For example, the function $f = {}^1\!/\!_2 r^2 - r^3 \sin(3\theta)$ in polar coordinates has a triangular contour $f = 1\big/54$. Translating this back to rectangular coordinates yields the Hamiltonian

$$H = \frac{1}{2}\left(x^2 + y^2\right) + y^3 - 3x^2 y. \tag{5.4}$$

As can be seen in Figure 5.2, H has three saddle equilibria $(x, y) = (\pm\sqrt{3}\big/6, 1\big/6)$, and $(0, -1\big/3)$ on the contour $H = 1\big/54$. There are three heteroclinic orbits connecting these saddles. When such a collection of heteroclinic orbits divides the plane into two regions we call it a *separatrix cycle*. ∎

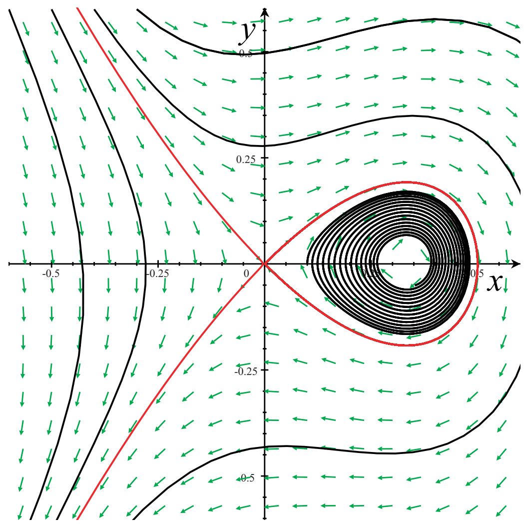

Figure 5.3. *Non-Hamiltonian system* (5.6) *with a homoclinic orbit. Here* $a = 1$.

The existence of a saddle connection is unusual for general ODEs in the plane; however, with some care we can construct examples that do have a connection.

Example: Given a Hamiltonian system with a homoclinic orbit, it is easy to construct a non-Hamiltonian system that has one as well; such an example was given in (4.45). More generally, the contour $H(x, y) = E$ is preserved by the differential equations

$$\begin{aligned}\frac{dx}{dt} &= \frac{\partial H}{\partial y} + (H(x, y) - E)g_1(x, y),\\ \frac{dy}{dt} &= -\frac{\partial H}{\partial x} + (H(x, y) - E)g_2(x, y)\end{aligned} \tag{5.5}$$

for *any* functions g_1 and g_2. If this contour contains a homoclinic orbit, then (5.5) will have a homoclinic orbit too. In the example (5.3), the homoclinic orbit was at $E = 0$; therefore, the system

$$\begin{aligned}\dot{x} &= y + H(x, y)x,\\ \dot{y} &= x - 3ax^2 + H(x, y)y,\end{aligned} \tag{5.6}$$

shown in Figure 5.3, still has the same homoclinic loop as the original Hamiltonian flow shown in Figure 5.1. Note that the origin is still a saddle. There are two more equilibria at $y^* = {}^1\!/_2 a{x^*}^4$ where x^* is a real root of the sixth-order polynomial $-4 + 12ax + a^2x^6$. For $a > 0$, the positive root of this polynomial is near the original center; however, this point is now a stable focus and attracts every point inside the homoclinic loop; see Exercise 2. ∎

5.3 Stable Manifolds

We can sometimes find W^u and W^s analytically even for the non-Hamiltonian case if the system of equations is a *skew product*; for example, if one of the equations of an ODE system in $\mathbb{R}^2$ is independent of the other. This kind of example seems special at first, but will prove to be of great use to us in the next section in the general proof of the stable manifold theorem.

Example: For example, suppose that $(x, y) \in \mathbb{R}^2$ and

$$\begin{aligned} \dot{x} &= -x, \\ \dot{y} &= y + g(x). \end{aligned} \tag{5.7}$$

Here, we will assume that g is C^1 and that $g(0) = 0$. The latter condition ensures that the origin is an equilibrium. The Jacobian of the origin is

$$Df(0) = \begin{pmatrix} -1 & 0 \\ Dg(0) & 1 \end{pmatrix}.$$

This matrix has eigenvalues $\lambda = \pm 1$ and so is hyperbolic. The unstable eigenvector is $v_u = (0, 1)^T$ so that the unstable subspace is the y-axis:

$$E^u = \{(x, y) : x = 0\}.$$

The second eigenvector is $v_s = (2, -Dg(0))^T$, so that the stable subspace is the line

$$E^s = \{(x, y) : Dg(0)x + 2y = 0\}.$$

Our goal is to find the stable and unstable sets of the origin. The ODEs are simple enough that the flow is easily obtained. Solving the x equation gives $x(t) = x_o e^{-t}$. Substituting this into the y equation yields a nonautonomous linear equation. We can use the integrating factor method (recall Exercise 2.17) to find

$$\frac{d}{dt}(e^{-t} y) = e^{-t} g(x_o e^{-t}) \quad \Rightarrow e^{-t} y(t) = y_o + \int_0^t e^{-s} g(x_o e^{-s}) ds.$$

Upon changing integration variables, setting $u = e^{-s}$, and putting the two solutions together, we obtain the expression for the flow:

$$\varphi_t \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x e^{-t} \\ y e^t + e^t \displaystyle\int_{e^{-t}}^1 g(xu) du \end{pmatrix}.$$

Since this is the general solution, we can find the set of points (x, y) that lie, for example, on the unstable manifold by asking which points have $\varphi_t(x, y) \to (0, 0)$ as $t \to -\infty$. This immediately implies that $x = 0$, since otherwise the first component is unbounded. In this case, since $g(0) = 0$, the second component becomes ye^t, which does approach 0. So we have shown that $W^u(0, 0)$ is simply the y-axis.

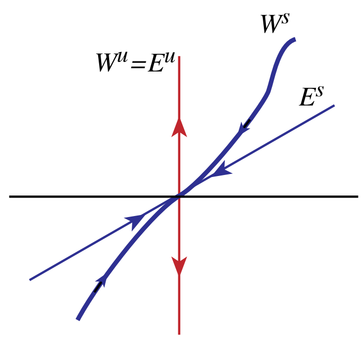

Figure 5.4. *Sketch of stable and unstable manifolds for* (5.7).

The stable set, $W^s(0, 0)$, is the set such that $\varphi_t(x, y) \to (0, 0)$ as $t \to \infty$. This means that x can be arbitrary, but y must be chosen specifically since we require

$$0 = \lim_{t\to\infty} y(t) = \lim_{t\to\infty} e^t \left(y + \int_{e^{-t}}^{1} g(xu)du \right).$$

We claim that for each x there is a solution of the form $y(x) = -\int_0^1 g(xu)du$. To see this, substitute it into the limit to obtain

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} e^t \left(\int_{e^{-t}}^{1} g(xu)du - \int_0^1 g(xu)du \right) = -\lim_{t\to\infty} e^t \left(\int_0^{e^{-t}} g(xu)du \right).$$

Since $g(0) = 0$ and $g \in C^0$, then for any ε, there is a δ such that $|g(xu)| < \varepsilon$ for all $|xu| < \delta$. If we choose t large enough so that $|x|\, e^{-t} < \delta$, then the magnitude of the integral is bounded by εe^{-t}. Since this is true for any ε, the limit is zero as required. Thus, we have shown that

$$W^s = \left\{ (x, y(x)) : \;\; y(x) = -\int_0^1 g(xu)du \right\}, \tag{5.8}$$

as sketched in Figure 5.4. For example, if

$$g(x) = -\sin x, \tag{5.9}$$

we can easily do the integral in (5.8) to obtain the function

$$y(x) = -\frac{1}{x}\int_0^x \sin(\xi)d\xi = \frac{1 - \cos x}{x}.$$

The phase portrait of this case is shown in Figure 5.5.

Note that W^s is tangent to E^s at the origin because its slope is

$$\left.\frac{dy}{dx}\right|_{x=0} = -\left.\int_0^1 Dg(xu)udu\right|_{x=0} = -\frac{1}{2}Dg(0),$$

which is precisely the slope of E^s. This tangency property will be generalized to the fully nonlinear case below. Since y is expressed as a function of x in (5.8) and each x determines

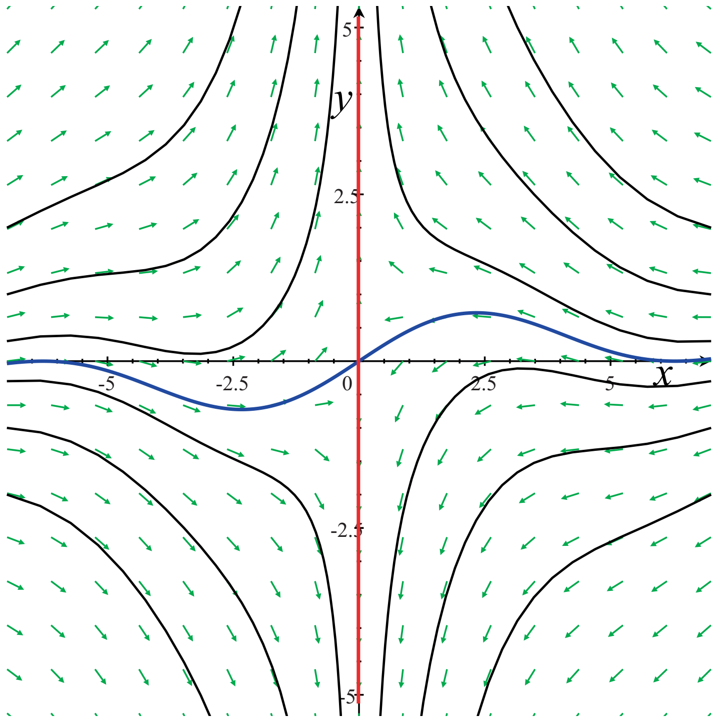

Figure 5.5. *Phase portrait for* (5.7) *with* $g(x)$ *given by* (5.9)*. Here the unstable manifold is the* y*-axis (red line) and the stable manifold is the blue curve. Several other trajectories are also shown.*

a unique point on E^s, the stable manifold is a graph over E^s. Finally, both W^u and W^s are smooth curves, that is, they are *manifolds*. ∎

In the construction of the manifolds in the example above, we noticed that W^s is a graph over E^s. To use this property for a general hyperbolic equilibrium, we define projection operators onto E^s and E^u. A projection is a linear operator $\pi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\pi \circ \pi = \pi$. We will define two projections π_u and π_s such that $\pi_u + \pi_s = id$; see Figure 5.6. These projections formalize the idea of finding components of a vector "along the eigenvectors." Recall from §2.6 that any vector can be written as a linear combination of generalized eigenvectors,

$$x = \sum_{j=1}^{n} c_j v_j .$$

In other words, there is a nonsingular matrix $P = [v_1, v_2, \ldots, v_n]$ such that $x = Pc$ and $c = P^{-1}x$. If the first k of these vectors span E^u, then the projections are given by

$$\pi_u(x) = \sum_{j=1}^{k} c_j v_j , \qquad \pi_s(x) = \sum_{j=k+1}^{n} c_j v_j .$$

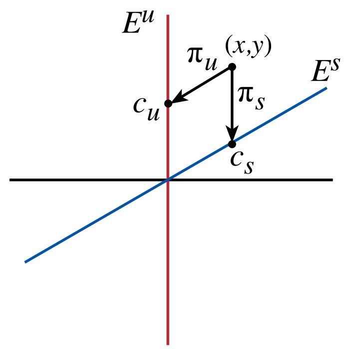

Figure 5.6. *Projections onto E^u and E^s.*

Example: For the system (5.7) $P = (v_u, v_s) = \begin{pmatrix} 0 & 2 \\ 1 & -Dg(0) \end{pmatrix}$, so that

$$\begin{pmatrix} c_u \\ c_s \end{pmatrix} = P^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2} \begin{pmatrix} Dg(0) & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} Dg(0)x + y \\ \frac{1}{2}x \end{pmatrix}.$$

Thus, the projection operators onto E^u and E^s are

$$\pi_u \begin{pmatrix} x \\ y \end{pmatrix} = c_u v_u = \begin{pmatrix} 0 \\ y + \frac{1}{2}x Dg(0) \end{pmatrix}, \quad \pi_s \begin{pmatrix} x \\ y \end{pmatrix} = c_s v_s = \begin{pmatrix} x \\ -\frac{1}{2}x Dg(0) \end{pmatrix}. \quad \blacksquare$$

With these examples under our belt, we proceed to develop a general understanding of the stable and unstable manifolds of a saddle equilibrium. We begin by restricting our study to a neighborhood of the equilibrium to construct the "local" manifolds.

5.4 Local Stable Manifold Theorem

In this section we will show that the stable and unstable sets of a hyperbolic equilibrium are actually smooth manifolds when the vector field is C^1. Suppose that x^* is a hyperbolic equilibrium with linearization $Df(x_o) = A$. We can always shift coordinates so that the equilibrium is at the origin by replacing $x \to x + x^*$, so that the equations take the form

$$\dot{x} = Ax + g(x), \tag{5.10}$$

where $g(x) = f(x + x^*) - Ax$ represents the nonlinear terms in the equation so that $g(0) = 0$ and $Dg(0) = 0$. Since A is hyperbolic, there is an $\alpha > 0$ such that $|\mathrm{Re}\lambda_i| > \alpha$ for all eigenvalues λ_i of A. The projection operators are $\pi_s : \mathbb{R}^n \to E^s$ and $\pi_u : \mathbb{R}^n \to E^u$. Note that since A leaves E^s and E^u invariant, it commutes with the projections

$$\pi_u A = A\pi_u \text{ and } \pi_s A = A\pi_s.$$

The same is true for the fundamental matrix e^{tA}. Moreover, the estimate (2.44) in §2.7 implies that there is a $K \geq 1$ such that

$$\begin{aligned} \left|e^{tA}\pi_s x\right| &\leq Ke^{-\alpha t}\left|\pi_s x\right|, \\ \left|e^{-tA}\pi_u x\right| &\leq Ke^{-\alpha t}\left|\pi_u x\right|, \end{aligned} \quad t \geq 0. \tag{5.11}$$

Our goal is to prove that the stable set W^s for (5.10) is a smooth manifold, and our main tool is the contraction-mapping theorem (what else!). The first step is to find the appropriate operator, T, and function space. To motivate the construction of T—which generalizes the operator (3.11) used to prove existence and uniqueness—we first study a simpler set of affine ODEs.

Lemma 5.2. *Consider the affine, nonautonomous initial value problem*

$$\dot{x} = Ax + \gamma(t), \quad \pi_s x(0) = \sigma \in E^s. \tag{5.12}$$

Suppose A is hyperbolic and $\gamma(t)$ is bounded and continuous for $t \geq 0$. Then the unique solution, $x(t;\sigma)$, of (5.12) *that is bounded for positive time is*

$$x(t) = e^{tA}\sigma + \int_0^t e^{(t-s)A}\pi_s\gamma(s)ds - \int_t^\infty e^{(t-s)A}\pi_u\gamma(s)ds. \tag{5.13}$$

The uniqueness of the solution (5.13) is surprising because only "half" of the initial conditions have been specified, the stable components σ. We will see that the assumption that x is bounded for $t > 0$ is enough to determine its unstable components.

Proof. The general solution of the forced linear equation can be obtained by the integrating factor method or the method of variation of parameters. To implement the latter, guess a solution of the form $x(t) = e^{tA}\xi(t)$. Substitute this into the ODE to obtain $\dot{\xi} = e^{-tA}\gamma(t)$, which can be solved trivially by integrating. If we specify the initial condition $x(\tau)$ at some arbitrary time $t = \tau$, the general solution to (5.12) has the form

$$x(t) = e^{(t-\tau)A}x(\tau) + \int_\tau^t e^{(t-s)A}\gamma(s)ds. \tag{5.14}$$

Our goal is to find a particular case of (5.14) that is bounded in forward time. We write $x(t) = \pi_u x(t) + \pi_s x(t)$ and consider these two projections separately.

First set $\tau = 0$ and take the stable projection of (5.14). Noting that $\pi_s x(0) = \sigma$, we obtain

$$\pi_s x(t) = e^{tA}\sigma + \int_0^t e^{(t-s)A}\pi_s\gamma(s)ds.$$

To show that this expression is bounded as $t \to \infty$, we use the assumption that γ is bounded, i.e., that there is a δ such that $|\gamma(s)| \leq \delta$ for all $s \geq 0$. Imposing the bound (5.11) then gives

$$\left|\int_0^t e^{(t-s)A}\pi_s\gamma(s)ds\right| \leq K\int_0^t e^{-(t-s)\alpha}\,|\pi_s\gamma(s)|\,ds \leq \frac{K}{\alpha}\delta.$$

Consequently, the stable projection of our solution is indeed bounded.

Projecting (5.14) onto the unstable space yields

$$\pi_u x(t) = e^{tA}\left(e^{-\tau A}\pi_u x(\tau) + \int_\tau^t e^{-sA}\pi_u\gamma(s)ds\right). \tag{5.15}$$

We must choose $\pi_u x(t)$ so that (5.15) remains bounded. Since the exponential $e^{tA}\pi_u$ generally grows without bound, a necessary condition is that the term in parenthesis in (5.15) limits to zero as $t \to \infty$, that is, if

$$e^{-\tau A}\pi_u x(\tau) = -\int_\tau^\infty e^{-sA}\pi_u \gamma(s)ds.$$

Since this is true for any τ, we can replace τ by t in this equation to obtain

$$\pi_u x(t) = -\int_t^\infty e^{(t-s)A}\pi_u \gamma(s)ds. \tag{5.16}$$

Substitution of (5.16) back into (5.15) gives an identity; therefore, (5.16) is a solution for the unstable projection. We now show that (5.16) is indeed bounded. The integral in (5.16) can be bounded using the bound (5.11) on $e^{\tau A}\pi_u$ for $\tau = t - s < 0$:

$$|\pi_u x(t)| = \left| \int_t^\infty e^{(t-s)A}\pi_u \gamma(s)ds \right| \le K \int_t^\infty e^{(t-s)\alpha} |\pi_u \gamma(s)|\, ds \le \frac{K}{\alpha}\delta.$$

Thus, (5.16) is both necessary and sufficient for the unstable projection being bounded.

Adding the stable and unstable projections gives the promised result (5.13). □

We now return to (5.10), where $\gamma(t)$ is replaced by the nonlinear function $g(x)$. If we similarly replace $\gamma(s)$ in integrand of (5.13) with $g(x(s))$, the resulting integral equation is satisfied by a solution of (5.10). Just as for the integral operator (3.11), which we used to prove existence and uniqueness, the new integral equation can be viewed as an operator on a suitable function space. Indeed we will show that this operator is a contraction map whose fixed point is the stable manifold of (5.10). Since g is nonlinear, we must restrict the analysis to a neighborhood of the equilibrium where g is sufficiently small; thus, we will only prove the existence of a "local" stable manifold, W^s_{loc}: the set of points on W^s that remain in some neighborhood of the equilibrium for all $t \ge 0$. The global stable manifold will be constructed from the local one in §5.5.

Theorem 5.3 (Local Stable Manifold). *Let A be hyperbolic, $g \in C^k(U)$, $k \ge 1$, for some neighborhood U of 0, and $g(x) = o(x)$ as $x \to 0$. Denote the linear stable and unstable subspaces of A by E^s and E^u. Then there is a $\tilde{U} \subset U$ such that local stable manifold of* (5.10),

$$W^s_{loc}(0) \equiv \left\{ x \in W^s(0) : \varphi_t(x) \in \tilde{U},\ t \ge 0 \right\},$$

is a Lipschitz graph over E^s that is tangent to E^s at 0. Moreover, $W^s(0)$ is a C^k manifold.

Since this is a rather long proof, we divide it into three parts. In the first part we prove that there is a unique, forward bounded solution for each point $\sigma \in E^s$ close enough to the origin. We then show in the second part that these solutions actually are on the stable manifold, since they are asymptotic to 0. In the final part of the proof, we show that these solutions lie on a smooth, Lipschitz graph.

***Proof* (*Part* 1).** By analogy with (5.13), define an operator $T : C^0(\mathbb{R}^+, \mathbb{R}^n) \to C^0(\mathbb{R}^+, \mathbb{R}^n)$ for a given point $\sigma \in E^s$ of A by

$$T(x)(t) = e^{tA}\sigma + \int_0^t e^{(t-s)A}\pi_s g(x(s))ds - \int_t^\infty e^{(t-s)A}\pi_u g(x(s))ds. \tag{5.17}$$

It is clear that if $x \in C^0(\mathbb{R}^+, \mathbb{R}^n)$, then so is $T(x)$. It is not hard to show that a sufficiently small, continuous fixed point of T, $x : \mathbb{R}^+ \to U$ is a C^1 solution of the ODE (5.10), call it $x(t; \sigma)$ (see Exercise 5).

We first show that T is a contraction map and therefore that the fixed point of T exists and is unique. To do this, define a closed subset of the function space $C^0(\mathbb{R}^+)$ by

$$V_\delta = \left\{x \in C^0(\mathbb{R}^+, \mathbb{R}^n) : \|x\| \le \delta\right\}, \tag{5.18}$$

where $\|x\|$ is the sup-norm (3.3). As discussed in §3.2, this space with the sup-norm is complete. Since $g(x) = o(x)$ as $x \to 0$ (recall §4.4), then for any $\varepsilon > 0$—no matter how small—there is a δ, such that when $x \in V_\delta$, then $|g(x(t)| \le \varepsilon\,|x(t)|$. Using the bounds (5.11) in (5.17) we obtain

$$|T(x)(t)| \le Ke^{-t\alpha}|\sigma| + K\varepsilon\int_0^t e^{-(t-s)\alpha}|x(s)|ds + K\varepsilon\int_t^\infty e^{(t-s)\alpha}|x(s)|ds \le K|\sigma| + 2\frac{K\varepsilon}{\alpha}\delta$$

for any for $t \ge 0$. The necessary bound $\|T(x)\| \le \delta$ can be satisfied by requiring, e.g.,

$$|\sigma| < \delta\big/2K \text{ and } \varepsilon \le \alpha\big/4K. \tag{5.19}$$

These requirements define the neighborhood

$$\tilde{U} = \left\{x :\ |g(x)| \le \frac{\alpha}{4K}\,|x|\right\} \cap U \tag{5.20}$$

that effectively defines δ, since ε can be made arbitrarily small for a sufficiently small δ

We now show that T is a contraction. Since $g \in C^1$, and $\|Dg(x)\| \le \varepsilon$ for $|x| \le \delta$, then (3.8) implies that $|g(x) - g(y)| \le \varepsilon\,|x - y|$ for $x, y \in B_\delta(0)$. Using this and (5.11) gives

$$|T(x) - T(y)| \le K\varepsilon\,\|x - y\|\left(\int_0^t e^{-(t-s)\alpha}ds + \int_t^\infty e^{(t-s)\alpha}ds\right) \le 2\frac{K\varepsilon}{\alpha}\,\|x - y\|\,.$$

Therefore, T is a contraction when $\varepsilon \le \alpha\big/4K$, which we already assumed, and the contraction-mapping theorem implies that T has a unique fixed point in V_δ. Since there is a unique fixed point $x(t; \sigma)$ for each $\sigma \in E^s$ providing $|\sigma| < \delta\big/2K$, the set $x(0; \sigma)$ is a graph over E^s. □

***Proof* (*Part* 2).** To show that $x(t; \sigma)$ is a point on the stable manifold, we must show it approaches zero as $t \to \infty$. Since is $x(t; \sigma)$ is a fixed point of T, we use (5.11) to bound it by

$$|x(t; \sigma)| \le Ke^{-\alpha t}\,|\sigma| + K\varepsilon\int_0^t e^{-\alpha(t-s)}\,|x(s; \sigma)|\,ds + K\varepsilon\int_t^\infty e^{\alpha(t-s)}\,|x(t; \sigma)|\,ds. \tag{5.21}$$

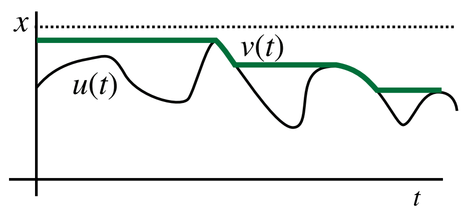

Figure 5.7. *Construction of the function* $v(t)$ *in* (5.23).

We assert that this implies that $x \to 0$ exponentially fast. To show this, we need a generalization of Grönwall's inequality (3.30). □

Lemma 5.4 (Generalized Grönwall). *Suppose* α, M, *and* L *are nonnegative,* $L < \alpha/2$, *and there is a nonnegative, bounded, continuous function* $u : \mathbb{R}^+ \to \mathbb{R}^+$ *satisfying*

$$u(t) \le e^{-\alpha t} M + L \int_0^t e^{-\alpha(t-s)} u(s) ds + L \int_t^\infty e^{\alpha(t-s)} u(s) ds; \tag{5.22}$$

then $u(t) \le \frac{M}{\beta} e^{-(\alpha - L/\beta)t}$, *where* $\beta = 1 - 2\frac{L}{\alpha}$.

Putting aside the proof of the lemma for the moment, note that it applies to the inequality (5.21) since we know that the fixed point $x(t;\sigma)$ is continuous. We set $u = |x(t;\sigma)|$, $L = K\varepsilon$, and $M = K|\sigma|$. Then $4K\varepsilon/\alpha \le 1$ implies that $\beta = 1 - 2K\varepsilon/\alpha \ge 1/2$, and letting $c \equiv 2K\varepsilon/\alpha \le 1/2$, we have $\frac{L}{\beta} = \frac{\alpha}{2}\frac{c}{1-c} \le \frac{\alpha}{2}$. So the hypotheses of Lemma 5.4 apply and give

$$|x(t;\sigma)| \le 2Ke^{-\alpha t/2} |\sigma| ,$$

implying that $x(t;\sigma) \to 0$ exponentially fast.

Proof of Lemma. By assumption u is bounded; therefore, we can define its supremum. Moreover, the function

$$v(t) = \sup_{s>t} u(s) \tag{5.23}$$

exists and is nonincreasing: $v(t) \le v(s)$ if $s \ge t$; see Figure 5.7. Since u is continuous, for any t there is a $T \ge t$ such that $v(t) = v(T)$ and thus from (5.22)

$$\begin{aligned} v(t) = u(T) &\le e^{-T\alpha} M + L \int_0^T e^{-\alpha(T-s)} u(s) ds + L \int_0^\infty e^{-\alpha s} u(T+s) ds \\ &\le e^{-T\alpha} M + L \int_0^t e^{-\alpha(T-s)} u(s) ds + L \int_t^T e^{-\alpha(T-s)} u(s) ds \\ &\quad + L \int_0^\infty e^{-\alpha s} u(T+s) ds \\ &\le e^{-T\alpha} M + L \int_0^t e^{-\alpha(T-s)} u(s) ds + 2\frac{L}{\alpha} v(t), \end{aligned}$$

where we have used the facts that $u(s) \le v(t)$ and $u(T+s) \le v(T) = v(t)$ to approximate the last two integrals. Rearranging this gives

$$\left(1 - 2\frac{L}{\alpha}\right) e^{\alpha t} v(t) \le e^{-\alpha(T-t)} M + L \int_0^t e^{-\alpha(T-t)} e^{\alpha s} u(s) ds.$$

Defining $z(t) = \beta e^{\alpha t} v(t)$, and noting that $e^{-\alpha(T-t)} \le 1$, we have

$$z(t) \le M + \frac{L}{\beta} \int_0^t z(s) ds.$$

This is of the form of Grönwall's lemma (3.30), so that $z(t) \le M e^{tL/\beta}$. Rewriting this in terms of $u(t) \le v(t)$ gives the promised result. □

***Proof* (*Part* 3).** It is relatively easily to see that the solutions $x(t;\sigma)$ lie on a Lipschitz graph, i.e., that the unstable components are Lipschitz functions of σ. To show this, consider $\pi_u x$ at two different σ values, subtract the fixed-point equations $x = T(x)$, and take the projections onto E^u. Using the fact that π_u annihilates σ, we obtain

$$|\pi_u \left(x(t;\sigma_1) - x(t;\sigma_2)\right)| \le K\varepsilon \int_t^\infty e^{(t-s)\alpha} \left|x(s;\sigma_1) - x(s;\sigma_2)\right| ds. \tag{5.24}$$

To evaluate this, we must also bound the difference in the integral, which we can do with the same integral equation:

$$\begin{aligned} |x(t;\sigma_1) - x(t;\sigma_2)| \le K e^{-\alpha t}|\sigma_1 - \sigma_2| &+ K\varepsilon \int_0^t e^{-(t-s)\alpha} \left|x(s;\sigma_1) - x(s;\sigma_2)\right| ds \\ &+ K\varepsilon \int_t^\infty e^{(t-s)\alpha} \left|x(s;\sigma_1) - x(s;\sigma_2)\right| ds. \end{aligned}$$

This is of the form (5.22), so the generalized Grönwall inequality yields

$$|x(t;\sigma_1) - x(t;\sigma_2)| \le 2K e^{-\alpha t/2}|\sigma_1 - \sigma_2|.$$

Consequently, $x(t;\sigma)$ is a Lipschitz function of σ. We can now use this bound in (5.24) to obtain

$$|\pi_u x(t;\sigma_1) - \pi_u x(t;\sigma_2)| \le \frac{4K^2\varepsilon}{3\alpha} e^{-\alpha t/2} \left|\sigma_1 - \sigma_2\right|,$$

giving the promised Lipschitz condition. □

Differentiability of the stable set is more difficult to prove. The basic principle we will use is the following generalization of Theorem 3.4, the contraction-mapping theorem: if a contraction map depends smoothly on parameters, its fixed points must as well.

Theorem 5.5 (Uniform Contraction Principle). *Let X and Y be closed subsets of two Banach spaces and let $T \in C^k(X \times Y, X)$, $k \ge 0$, be a uniform contraction map.*[32] *Then there is a unique fixed point, $x(y) = T(x(y), y)$, where $x(y) \in X$ is a C^k function of $y \in Y$.*

[32]This means that the contraction constant $c < 1$ is independent of y and that $T(x;y)$ is a uniformly C^k function of y.

Delaying the proof of this theorem for the moment, note that it gives the promised result. It applies to our map T because when g is C^k, the fixed point, $x(t;\sigma)$ is also C^k in both t and σ. It also implies the tangency of W^s to E^s, since the Jacobian matrix obtained from differentiating x with respect to σ at $\sigma = 0$ is

$$D_\sigma x(t;0) = e^{tA}\pi_s + \left(\int_0^t ds e^{(t-s)A}\pi_s - \int_t^\infty ds e^{(t-s)A}\pi_u\right) Dg\,(x(s;0))\, D_\sigma x(s;0) = e^{tA}\pi_s,$$

where we have used the facts that $x(s;0) = 0$ is the unique fixed point when $\sigma = 0$ and that $Dg(0) = 0$. Thus, for any v, $D_\sigma x(t;0)v \in E^s$, so that W^s is tangent to E^s.

***Proof of Theorem* 5.5.** Let $\| \ \|$ denote the norms on both X, and Y. Since T is a uniform contraction, there is a constant c such that $0 < c < 1$ and $\|T(x;y) - T(\xi,y)\| \le c\,\|x-\xi\|$ for all $x, \xi \in X$, and $y \in Y$. Moreover, the contraction mapping theorem, Theorem 3.4, implies that for each y there is a unique fixed-point $x(y) = T(x(y);y)$.

Suppose first that T is uniformly C^0. We will show that the fixed point, $x(y)$, is uniformly continuous. The fixed-point equation and triangle inequality imply that for any $h \in Y$

$$\begin{aligned}
\|x(y+h) - x(y)\| &= \|T(x(y+h); y+h) - T(x(y);y)\| \\
&\le \|T(x(y+h); y+h) - T(x(y); y+h)\| \\
&\quad + \|T(x(y); y+h) - T(x(y);y)\| \\
&\le c\,\|x(y+h) - x(y)\| + \|T(x(y); y+h) - T(x(y);y)\|\,.
\end{aligned}$$

Since T is uniformly continuous in y,for every ε there is an h such that $\|T(x; y+h) - T(x,y)\| \le \varepsilon$; using this value of h, the previous inequality gives

$$\|x(y+h) - x(y)\| \le \frac{\varepsilon}{1-c}$$

for any ε. This shows that x is uniformly continuous, since c and ε are independent of y.

It is much more difficult to prove smoothness; we will consider only the case $k = 1$. Suppose that T is uniformly C^1. If the fixed point $x(y) = T(x(y);y)$ were differentiable, then its derivative would obey the relation

$$D_y x(y) = D_x T(x(y);y) D_y x(y) + D_y T(x(y);y). \tag{5.25}$$

Replace $D_y x$ by a linear operator $M : X \to X$ and think of this equation as a linear system for an unknown M:

$$(I - D_x T(x(y);y))\, M = D_y T(x(y);y). \tag{5.26}$$

This system has a unique solution if the left-hand side is nonsingular.[33] This follows since $\|D_x T\| \le c < 1$; see Exercise 6. Now we must show that this $M(y)$ is really $D_y x$. Define

$$\xi(h) \equiv x(y+h) - x(y) = T(x(y) + \xi, y+h) - T(x(y);y).$$

[33] Equation (5.25) can also be thought of as a contraction map on $D_y x$ and so has a unique solution.

Combining this with (5.26) gives

$$(I - D_x T(x(y); y))\,(\xi(h) - M(y)h) = \Delta(\xi, h),$$
$$\Delta(\xi, h) \equiv T(x(y) + \xi; y + h) - T(x(y); y) - D_x T(x(y); y)\xi - D_y T(x(y); y)h.$$

If we can show that $\|\Delta\| \to 0$ as $\|h\| \to 0$, then because $I - D_x T$ is nonsingular, we would have $\xi(h) - Mh \to 0$, which would imply that $x(y)$ is differentiable with derivative M.

Since T is C^1, for any ε there is a δ such that when $|h| < \delta$ and $|\xi(h)| < \delta$, we have

$$\|\Delta(\xi, h)\| < \varepsilon\,(\|\xi(h)\| + \|h\|)\,. \tag{5.27}$$

This is not quite good enough since we do not have $\xi = O(h)$ yet. However, this can be obtained using the definition of Δ, which implies

$$\xi(y) = D_x T(x(y); y)\xi + D_y T(x(y); y)h + \Delta.$$

Using the bounds on $D_x T$ and Δ we obtain

$$\|\xi(h)\| \le c\,\|\xi(h)\| + \left\|D_y T(x(y); y)h\right\| + \varepsilon\,(\|\xi(h)\| + \|h\|) \quad \Rightarrow$$
$$\|\xi(h)\| \le \frac{\left\|D_y T(x(y); y)h\right\| + \varepsilon\,\|h\|}{1 - c - \varepsilon} \le C\,\|h\|\,,$$

providing $\varepsilon < 1 - c$. Putting this back into (5.27) gives

$$\|\Delta(\xi, h)\| \le \varepsilon\,(C + 1)\,\|h\|\,.$$

Therefore $\|\Delta\| \to 0$ as $\|h\| \to 0$.

Showing that x is C^k for $k > 1$ requires an additional inductive step. □

This completes, as well, our rather lengthy proof of Theorem 5.3. □

Example: The two-dimensional system

$$\begin{aligned} \dot{x} &= 2x + y^2, \\ \dot{y} &= -2y + x^2 + y^2 \end{aligned} \tag{5.28}$$

has a saddle at the origin with a diagonal Jacobian $Df(0, 0) = \mathrm{diag}(2, -2)$. Consequently, the linear spaces are $E^u = \mathrm{span}(1, 0)^T$ and $E^s = \mathrm{span}(0, 1)^T$ with the corresponding projection matrices

$$\pi_u = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \pi_s = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

These exemplify the general property $\pi_u + \pi_s = I$. Given a point $\sigma = (0, \sigma_y) \in E^s$, the operator (5.17) becomes

$$T(x) = \begin{pmatrix} 0 \\ e^{-2t}\sigma_y \end{pmatrix} + \begin{pmatrix} -e^{2t}\int_t^\infty e^{-2s} y^2(s)ds \\ e^{-2t}\int_0^t e^{2s}\left(x^2(s) + y^2(s)\right)ds \end{pmatrix}.$$

According to Theorem 5.3, we can begin with any function in V_δ providing δ is small enough. The crucial estimate is that $|g(x)| < \varepsilon\,|x|$, for $|x| < \delta$. For the example, $|g(x)| \le \sqrt{2}\delta^2$, so we may set $\delta = \varepsilon/\sqrt{2}$. Since $Df(0, 0)$ is diagonal with $|\lambda| = 2$, we may set $K = 1$ and

$\alpha = 2$ so the requirements (5.19) become

$$\varepsilon < \frac{1}{2} \text{ and } \delta < \frac{1}{2\sqrt{2}}.$$

Beginning with the initial guess $(x_0(t), y_0(t)) = (0, 0)$, clearly in V_δ, the first two iterates of T are

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = T(x_0, y_0) = \begin{pmatrix} 0 \\ e^{-2t}\sigma_y \end{pmatrix},$$

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = T(x_1, y_1) = \begin{pmatrix} -\frac{1}{6}e^{-4t}\sigma_y^2 \\ \\ e^{-2t}\sigma_y + \frac{1}{2}e^{-2t}\left(1 - e^{-2t}\right)\sigma_y^2 \end{pmatrix}.$$

Note that the approximate solutions do indeed limit to the origin as $t \to \infty$. To obtain a picture of the stable manifold, it is sufficient to plot the curve as a function of the initial point at any value of t, say, for example, at $t = 0$. In this case we have a parametric representation of the approximate stable manifold:

$$W^s_{loc}(0, 0) \approx \left\{(x_2(y), y)\,;\, x_2(y) = -\frac{1}{6}y^2,\ \ |y| < \frac{1}{2}\delta\right\}.$$

The next iterate gives an improved curve $x(y)$

$$x_3(y) = -\frac{1}{6}y^2\left(1 + \frac{1}{4}y + \frac{1}{240}y^2\right). \tag{5.29}$$

A plot of this curve, along with some representative trajectories, is shown in Figure 5.8. Note that the approximate manifold fails to capture the behavior near the spiral focus at $(-0.931, 1.364)$. ∎

5.5 Global Stable Manifolds

The stable manifold theorem implies that there is a neighborhood of a hyperbolic equilibrium for which the local stable manifold, W^s_{loc}, is a smooth submanifold of $\mathbb{R}^n$ with the same dimension as the stable subspace, E^s. On the other hand, the global stable set consists of *all* points that eventually limit on the equilibrium in forward time. As Lemma 5.1 implies, W^s is an invariant set: if $z \in W^s(x^*)$, then so are all the points on its orbit, $\varphi_t(z)$. Moreover, since every point on $W^s(x^*)$ must eventually stay in an arbitrarily small neighborhood x^*, the forward orbit of every point in W^s must eventually land in W^s_{loc}. Consequently, if we extend the local stable manifold by allowing each point to flow backward, we obtain the global stable set:

$$W^s = \left\{\varphi_t(x) :\ x \in W^s_{loc},\ \ t \in \mathbb{R}\right\}.$$

Since W^s_{loc} is smooth, and the orbits are smooth functions of time, the extension of W^s_{loc} for any *finite* value of t is as smooth as the vector field. However, it is not obvious that the set W^s defined for *all* t is quite so nice. The question that we seek to answer here is, how "nice" is W^s?

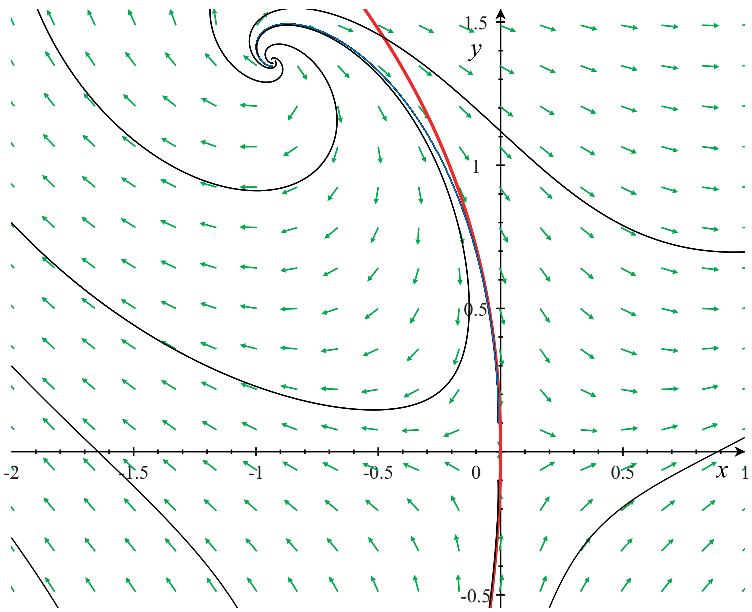

Figure 5.8. *Phase portrait of* (5.28) *and its approximate stable manifold* (5.29).

To discuss the structure of W^s, we briefly pause to consider several properties of maps from one space to another. Our goal is to define the concept of "embedding," which is, loosely speaking, what we think of when we imagine a smooth surface.

Mathematically, a relation of the form $g : M \to N$ that maps one space into another defines a surface—we say g is a *map*. So that it is *possible* for g to be one-to-one, we will require that $m = \dim(M) \leq n = \dim(N)$.

Example: Consider the map $g : \mathbb{S}^1 \to \mathbb{R}^2$ defined by $g(\theta) = (x(\theta), y(\theta)) = (2\cos\theta, \sin\theta)$. This is a mapping of a circle represented by the points $\theta \in [0, 2\pi)$ into $\mathbb{R}^2$ represented by points (x, y). Geometrically, the map describes an ellipse. Alternatively, the map $g(\theta) = (\sin(2\theta), \sin\theta)$ describes a figure eight; see Figure 5.9. Both are maps of the circle into the plane, but the latter map is not one-to-one. ∎

Both maps in the example are locally smooth in the sense that each component of g is a C^1 function. The Jacobian derivative of the map g, at a point $x \in M$, $Dg(x)$, is a matrix of dimension $n \times m$; it takes a vector v of dimension m and gives a new vector $w = Dg(x)v$ attached to the point $g(x) \in N$.[34] Indeed, this vector is tangent to the surface $g(M)$, and the range of $Dg(x)$ corresponds to the tangent plane to the surface. If the rank of $Dg(x)$ is m for all x, then the tangent planes are everywhere m-dimensional. Both maps in the first example have this property: since the derivative is a nonzero vector for all θ, $\text{rank}(Dg(\theta)) = 1$.

[34] Actually, $Dg(x)$, is a map from the tangent space of M to the tangent space of N. Thus, $Dg(x)v$ is a tangent vector, a point in the tangent space $TN_{g(x)}$. If $N = \mathbb{R}^n$, then we can identify the tangent space with $\mathbb{R}^n$.

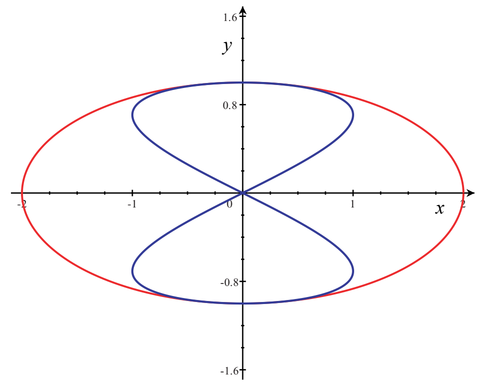

Figure 5.9. *Two maps of the circle into the plane.*

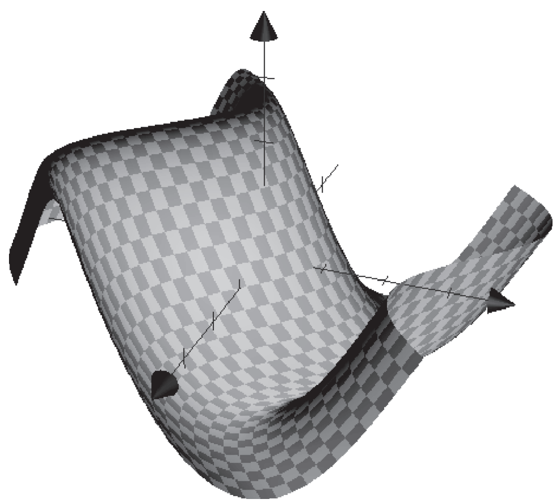

Figure 5.10. *Immersion* (5.30) *into* $\mathbb{R}^3$.

Example: The map $g : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$g(s,t) = (\cos t - s, \sin s - t, 2\sin t) \tag{5.30}$$

gives the surface shown in Figure 5.10. Its two tangent vectors are the columns of Dg, $v_1 = (-\sin t, -1, 2\cos t)^T$ and $v_2 = (-1, \cos s, 0)^T$. Since $v_1 \times v_2 \neq 0$, these vectors are never parallel and they define a two-dimensional plane tangent to the surface for each (s,t). ∎

A map with this property is an

▷ *immersion*: A C^1 map $g : M \to N$ is an immersion if $\operatorname{rank}(Dg) = \dim(M)$.

An immersion is locally a smooth surface.

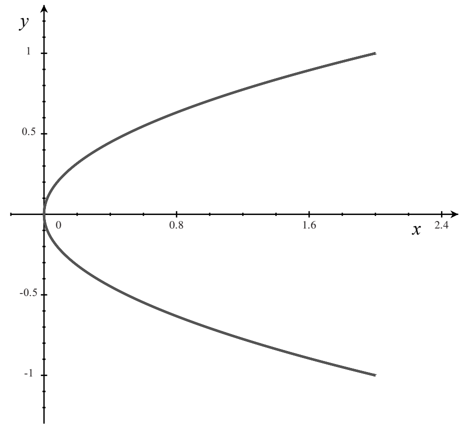

Figure 5.11. *Singular map* (5.31).

Example: Consider the map $g : \mathbb{R}^1 \to \mathbb{R}^2$ given by

$$g(t) = (1 + \cos(2t), \cos t)\,. \tag{5.31}$$

The rank of $Dg(t) = -(2\sin(2t), \sin t)$ is 1 except where it vanishes, i.e., when $t = n\pi$. The curve (5.31) has a cusp at these points, as shown in Figure 5.11. Consequently, it fails to be an immersion. ∎

The global stable manifold is easily seen to be an immersion:

Lemma 5.6. *Let f be a C^1 vector field on $\mathbb{R}^n$ with hyperbolic equilibrium at the origin having a k-dimensional stable space E^s. Then $W^s(0)$ is a k-dimensional immersion.*

Proof. Let the local stable manifold be defined by the map g: $g : E^s \to \mathbb{R}^n$ where $g(\sigma) = x(0; \sigma)$. The stable manifold theorem implies that W^s is an immersion since it defines a smooth Lipschitz graph over E^s. Hence, the rank of Dg is k. Each neighborhood of the global stable manifold can be obtained by flowing a region on W^s_{loc} backward in time for some fixed time. Thus for any neighborhood of W^s, we can consider the set of points defined by the map $h(\sigma) = \varphi_t(g(\sigma))$. This is smooth since φ is a smooth function of its arguments according to Theorem 3.15. Moreover, the derivative of this map is

$$Dh = D\varphi_t(g(\sigma))Dg(\sigma),$$

which has rank k since the matrix $\Phi = D\varphi_t$ solves the linearized differential equation (4.50) with initial condition $\Phi(0) = I$ and therefore is a nonsingular matrix. □

Even though an immersion is smooth, it may cross itself. For example, the figure-eight curve in Figure 5.9, though an immersion, is not one-to-one since both $\theta = 0$ and π are mapped to the origin. Even if we eliminate this problem by requiring that an immersion be one-to-one, there can be problems, as follows.

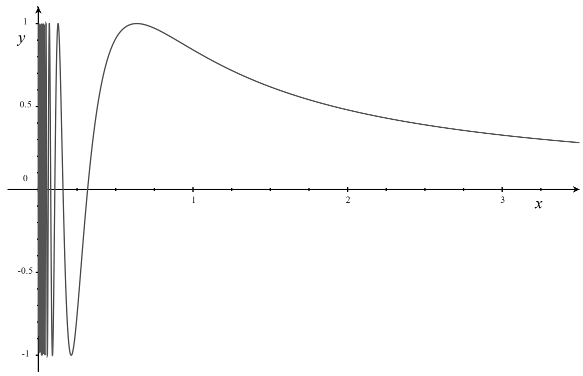

Figure 5.12. *The topologist's sine curve.*

Example: Consider the immersion $g : \mathbb{R} \to \mathbb{T}^2$ given by $g(t) = (t \bmod 1, \nu t \bmod 1)$, where ν is irrational. This is smooth and one-to-one but gives a dense line on the torus (see §7.1)—not what one would think of as a submanifold. ■

Example: The topologist's sine curve is the map $g : \mathbb{R}^+ \to \mathbb{R}^2$ defined by $g(t) = \left(1/t, \sin t\right)$. This curve is an immersion since $Dg \neq 0$ and is one-to-one. However, as $t \to 0$, the curve has infinitely many oscillations and accumulates upon the interval $[-1, 1]$ on the y-axis, as can be seen in Figure 5.12. ■

We will see later that the global stable manifold can have this accumulation problem: indeed, this is one of the indications of chaos. A map that does not have this pathology is called a

> ▷ *proper map*: A map $g : M \to N$ is proper if the preimage of every compact set in N is compact in M.

Example: The topologist's sine curve of Figure 5.12 is not proper because any neighborhood of the origin in $\mathbb{R}^2$ has a preimage consisting of infinitely many intervals of t in $(0, \infty)$. ■

We finally arrive at the ultimate definition of a "nice" map:

> ▷ *embedding*: A map $g : M \to N$ is an embedding if it is a one-to-one, proper immersion.

Of our examples above, only the ellipse and the map (5.30) are embeddings. However, any finite piece of W^s is an embedding, as follows from the next theorem.

Theorem 5.7. *If $g : M \to N$ is a C^1, one-to-one immersion, and both M and N are compact then it is automatically proper.*

Proof. Consider a compact subset $U \subset N$. Since U is closed, its complement is open. Since g is continuous, the preimage of any open set is open, and thus the preimage of U is the complement of an open set. Therefore, $g^{-1}(U)$ is closed and must be compact since it is a subset of the compact set M. So g is proper. □

5.6 Center Manifolds

Linear systems are classified according to their generalized eigenspaces, E^s, E^u, and E^c. The most important distinction was made between hyperbolic systems, where E^c is empty, and nonhyperbolic systems. We now begin our study of the behavior of a system with a nonhyperbolic fixed point—that is, for cases where E^c is not empty. This study will continue in Chapter 6 for the planar case and also will be a major focus of bifurcation theory in Chapter 8.

In the nonhyperbolic case it is still possible to construct stable and unstable manifolds at the fixed point for the hyperbolic directions. Moreover, the nonhyperbolic part of the dynamics can be reduced to a system of ODEs with the same dimension as the center subspace of the linear system. This is based on the following generalization of the stable manifold theorem.

Theorem 5.8 (Center Manifold). *Suppose that f is a C^k vector field, $k \geq 1$, with a fixed point at the origin. Let the eigenspaces of $Df(0) = A$ be written $E^u \oplus E^c \oplus E^s$. Then there is a neighborhood of the origin in which there exist C^k invariant manifolds: the local stable manifold, W^s_{loc}, tangent to E^s, on which $|x(t)| \to 0$ as $t \to \infty$, the local unstable manifold W^u_{loc}, tangent to E^u, on which $|x(t)| \to 0$ as $t \to -\infty$, and a local center manifold W^c, tangent to E^c.*

The proof of this theorem is more complicated than the stable manifold theorem; see (Carr 1981; Chicone 1999; Chow and Hale 1982; Hirsch, Pugh, and Shub 1977).

Note that this theorem does not state that the manifolds are unique, nor does it say that the manifolds are the only sets that have the proper asymptotic behavior. This is to be contrasted with the stable-manifold theorem for hyperbolic equilibria, which asserts that the local stable and unstable manifolds are unique and that they generate the global manifolds.

Example: Consider the skew-product system

$$\begin{aligned} \dot{x} &= x^2, \\ \dot{y} &= -y. \end{aligned} \tag{5.32}$$

Here, the linearization of the equilibrium at the origin has eigenvalues $\lambda = 0$ and -1, so the stable space E^s is the y-axis and the center space E^c is the x-axis. It is clear that the local stable manifold is the y-axis, since this is tangent to E^s and every point on the y-axis limits to the origin. We are tempted to say that W^c_{loc} is the x-axis, and this is certainly an acceptable center manifold: it is clearly an invariant set and is tangent to E^c. However, if we solve the equation for the phase curves, by dividing the y equation by the x equation, we obtain

$$\frac{dy}{dx} = -\frac{y}{x^2} \quad \Rightarrow y(x) = ce^{x^{-1}}.$$

When $x < 0$, each of these curves is asymptotic to the origin and is tangent to the x-axis (in fact, the function $y(x)$ has all derivatives zero at $x = 0^-$). So we could define a center manifold by

$$W^c_{loc}(0,0) = \left\{ (x, y) : y = \begin{bmatrix} ce^{x^{-1}} & x < 0 \\ 0 & x \geq 0 \end{bmatrix} \right\} \tag{5.33}$$

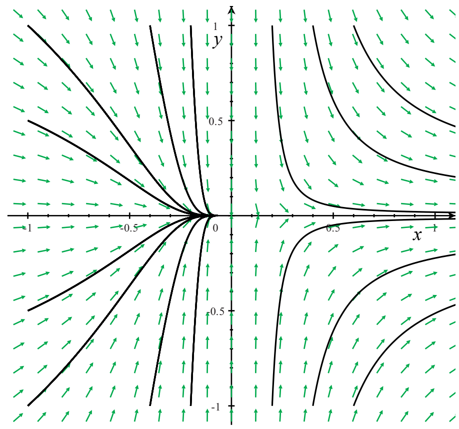

Figure 5.13. *Phase portrait of* (5.32).

for *any* value of c. There is a one-parameter family of possible center manifolds; see Figure 5.13! This example shows that the center manifold is *not* unique.

The example also has another pathology: though the local stable manifold is the y-axis, the *global* stable set—namely, the set of points that are asymptotic to the origin—is the left half-plane $W^s(0,0) = \{(x, y) : x \le 0\}$. Similarly, the *global* unstable set is the positive x-axis $W^u(0,0) = \{(x, 0) : x > 0\}$ since this set is asymptotic to the origin as $t \to -\infty$. ∎

In the example, the center manifold was not unique; nevertheless, every choice of c in (5.33) gives a curve with the *same* power series expression, namely, $y(x) = 0 + 0x + 0x^2 + \cdots$. Consequently, as far as the power series is concerned, there is a unique center manifold, the x-axis. Indeed, whenever f is C^∞, there is a unique power series expression for a center manifold.

This series can be easily determined by looking for functions corresponding to a graph that is tangent to E^c and demanding that the resulting surfaces are invariant. It is most convenient to do this by preparing the system so that the linear matrix breaks into blocks corresponding to the stable, unstable, and center subspaces. To do this, write the system as $\dot{\xi} = A\xi + g(\xi)$, where $g = o(\xi)$ represents the nonlinear terms. As we saw in §2.6, the matrix P of generalized eigenvectors transforms A to block diagonal form: $P^{-1}AP = J$, where

$$J = \begin{pmatrix} C & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & U \end{pmatrix}.$$

Here, C, S, and U are square matrices representing the center, stable, and unstable dynamics; they are diagonal only if A is semisimple. Then we define new coordinates $\eta = P^{-1}\xi$, so that

$$\begin{aligned} \dot{\eta} &= P^{-1}A\xi + P^{-1}g(\xi) = P^{-1}AP\eta + P^{-1}g(P\eta) \\ &= J\eta + P^{-1}g(P\eta). \end{aligned}$$

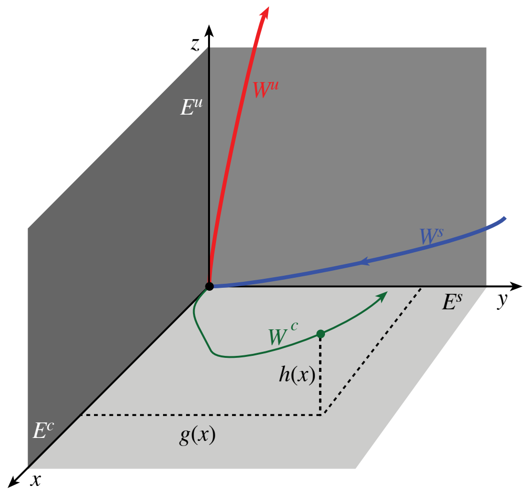

Figure 5.14. *Stable, unstable, and center manifolds.*

Now set $\eta = (x, y, z)$, where $\dim(x) = \dim(E^c)$, $\dim(y) = \dim(E^s)$, and $\dim(z) = \dim(E^u)$. In terms of the three subsets of variables, the ODEs now take the form

$$\begin{aligned} \dot{x} &= Cx + F(x, y, z), \\ \dot{y} &= Sy + G(x, y, z), \\ \dot{z} &= Uz + H(x, y, z). \end{aligned} \tag{5.34}$$

Since the local center manifold W^c is a graph over E^c, we can define it using two maps $g : E^c \to E^u$,and $h : E^c \to E^u$, so that on W^c we have $y = g(x)$ and $z = h(x)$ (see Figure 5.14), that is,

$$W^c = \{(x, g(x), h(x))\}.$$

The manifold begins at the origin, thus both $h(0) = g(0) = 0$; moreover, the manifold must be tangent to E^c, so $Dh(0) = Dg(0) = 0$. Finally, W^c must be invariant, so that if $(x, y, z) \in W^c$, then so is $\varphi_t(x, y, z)$. This means that the vector field $(\dot{x}, \dot{y}, \dot{z})$ must be in the tangent space of W^c. To compute this we insist that the flow lies on W^c, so that $y(t) = g(x(t))$ and $z(t) = h(x(t))$. Consequently, the derivatives of these functions must also match:

$$\dot{y} = Dg(x)\dot{x}, \quad \dot{z} = Dh(x)\dot{x}.$$

Putting this into (5.34) gives a system of PDEs that can be used to determine g and h:

$$\begin{aligned} Sg(x) + G(x, g(x), h(x)) &= Dg(x)\,(Cx + F(x, g(x), h(x)))\,, \\ Uh(x) + H(x, g(x), h(x)) &= Dh(x)\,(Cx + F(x, g(x), h(x)))\,. \end{aligned} \tag{5.35}$$

These PDEs can be solved order by order for the power series of h and g—see the examples below.

The dynamics on the center manifold are given by the equation for x upon restricting y and z to the manifold:

$$\dot{x} = Cx + F(x, h(x), g(x)). \tag{5.36}$$

That this equation describes the local dynamics on W^c follows from a generalization of the Hartman–Grobman theorem of §4.8.

Theorem 5.9 (Nonhyperbolic Hartman–Grobman). *Suppose* (5.34) *is a* C^1 *vector field with fixed point at the origin, that all the eigenvalues of* C *have zero real part, that* S *is a contracting and* U *is an expanding hyperbolic matrix, and that* $F, G, H = o(x, y, z)$. *Then there is a neighborhood* N *of the origin such that* $W^c_{loc} = \{(x, h(x), g(x)) : x \in E^c\} \cap N$ *and the dynamics in* N *is topologically conjugate to the system*

$$\begin{aligned} \dot{x} &= Cx + F(x, h(x), g(x)), \\ \dot{y} &= Sy, \\ \dot{z} &= Uz. \end{aligned} \tag{5.37}$$

Thus, the topological type of a nonhyperbolic fixed point is determined by the flow on the center manifold.

We now give several examples of the formal solution of the PDEs (5.35) order by order in the power series for the functions g and h.

Example: A two-dimensional system with a single zero eigenvalue has the block diagonal form

$$J = \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix}.$$

Here the center matrix is the 1×1 matrix $C = (0)$, and (taking $\lambda > 0$) the unstable matrix is $U = (\lambda)$. Therefore, the linear spaces are $E^c = \text{span}(1, 0)^T$ and $E^u = \text{span}(0, 1)^T$. For example, consider the C^∞ system

$$\begin{aligned} \dot{x} &= x^2 - z^2, \\ \dot{z} &= \lambda z + x^2. \end{aligned} \tag{5.38}$$

Following the general theory, we suppose that the local center manifold is $W^c_{loc}(0, 0) = \{(x, h(x)) : x \in \mathbb{R}\}$, where $h(0) = Dh(0) = 0$. Thus, the power series for h has the form $h(x) = \alpha x^2 + \beta x^3 + \cdots$. Putting this into (5.35) gives

$$\lambda\left(\alpha x^2 + \beta x^3 + \cdots\right) + x^2 = \left(2\alpha x + 3\beta x^2 + \cdots\right)\left(x^2 - \left(\alpha x^2 + \beta x^3 + \cdots\right)^2\right).$$

The lowest degree terms in this equation are quadratic and require that $\lambda\alpha + 1 = 0$. This determines α. The cubic terms give the equation $\lambda\beta = 2\alpha$, which determines β. After some algebra we find that

$$h(x) = -\frac{x^2}{\lambda} - 2\frac{x^3}{\lambda^2} - 6\frac{x^4}{\lambda^3} - 22\frac{x^5}{\lambda^4} - 96\frac{x^6}{\lambda^5} + \cdots.$$

The resulting curve $z = h(x)$ is shown in Figure 5.15. This result can be inserted into the differential equation for x, (5.38), to give the center manifold dynamics

$$\dot{x} = x^2 - \frac{x^4}{\lambda^2} - 4\frac{x^5}{\lambda^3} - 16\frac{x^6}{\lambda^4} \cdots. \tag{5.39}$$

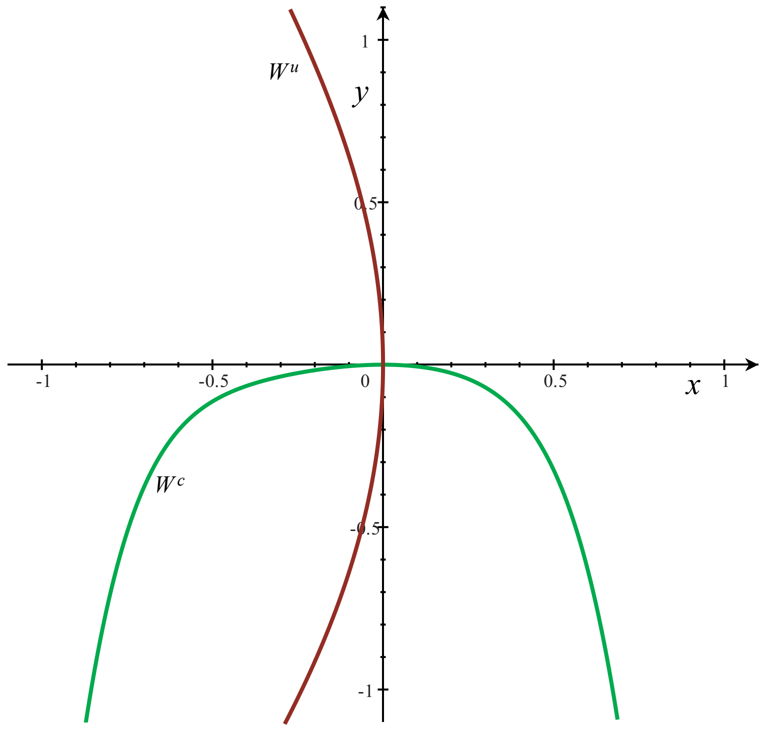

Figure 5.15. *Center and unstable manifolds for* (5.38) *through sixth order for* $\lambda = 2$.

This implies that $\dot{x} > 0$ when x is nonzero and small, which shows that on the center manifold the point $x = 0$ is "semistable"; see Figure 5.16.

The unstable manifold can be similarly found. If we let $x = g(z) = \alpha z^2 + \beta z^3 + \cdots$ and substitute this into the equation $\dot{x} = Dg(z)\dot{z}$, we obtain (after some algebra)

$$g(z) = -\frac{z^2}{2\lambda} + \frac{z^4}{16\lambda^2} + \frac{z^5}{20\lambda^4} - \frac{z^6}{96\lambda^5} + \cdots .$$

The curve $x = g(z)$ is shown in Figure 5.15.

According to Theorem 5.9, we have shown that (5.38) is conjugate to the system

$$\begin{aligned} \dot{x} &= x^2 - \frac{x^4}{\lambda^2} + \cdots , \\ \dot{z} &= z. \end{aligned}$$

If we compare the dynamics that we have found with a numerical solution of (5.38), see Figure 5.17, we see that the center and unstable manifolds prominently appear—note that the motion near the origin for decreasing t appears to rapidly compress along the unstable manifold (as e^{-t}) and then move more slowly along the center manifold toward the origin.

The system (5.38) has two additional fixed points, a saddle at $(\lambda, -\lambda)$ and a spiral sink at $(-\lambda, \lambda)$. The phase plane shows that the right branch of the center manifold appears

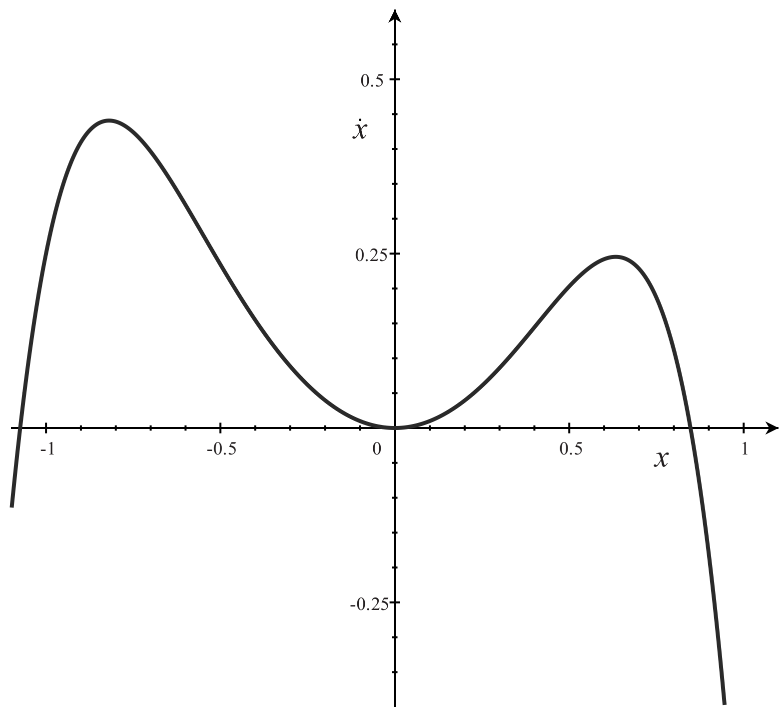

Figure 5.16. *The vector field* (5.39) *as a function of x on the local center manifold for $\lambda = 2$.*

to coincide with the stable manifold of the saddle. The spiral sink traps the bottom branch of $W^u(0)$. ∎

Example: Consider the three-dimensional system

$$\begin{aligned} \dot{x}_1 &= -x_2 + x_1 y, \\ \dot{x}_2 &= x_1 + x_2 y, \\ \dot{y} &= -y - x_1^2 - x_2^2 + y^2, \end{aligned} \qquad Df(0) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{5.40}$$

Here, Df is already in the normal form, and we can immediately see that $E^c = \{(x_1, x_2, 0)\}$ and $E^s = \{(0, 0, y)\}$. Again, look for solutions that are tangent to the center space, so that $W^c = \{(x_1, x_2, h(x_1, x_2))\}$. As before, assume a power series for $h(x) = \alpha x_1^2 + \beta x_1 x_2 + \gamma x_2^2 + \cdots$. Requiring that $y = h(x)$ is an invariant manifold, (5.35), gives

$$\begin{aligned} \dot{y} = Dh(x)\dot{x} &= \frac{\partial h}{\partial x_1}\dot{x}_1 + \frac{\partial h}{\partial x_2}\dot{x}_2, \\ -\alpha x_1^2 - \beta x_1 x_2 - \gamma x_2^2 - x_1^2 - x_2^2 + \cdots &= (2\alpha x_1 + \beta x_2 + \cdots)(-x_2 + \cdots) \\ &\quad + (\beta x_1 + 2\gamma x_2 + \cdots)(x_1 + \cdots) \end{aligned}$$

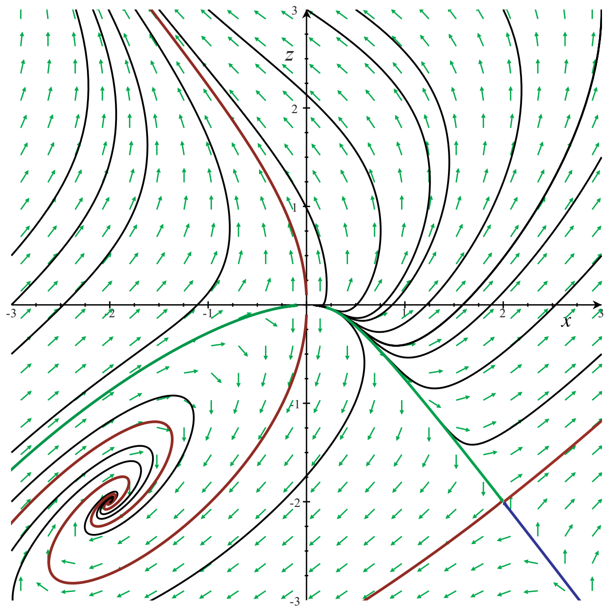

Figure 5.17. *Phase plane of* (5.38) *for* $\lambda = 2$.

to quadratic order. Collecting the terms in x_1^2, x_2^2, and $x_1 x_2$ gives three equations for the three unknowns α, β, and γ. These can be written as a single linear system:

$$\begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & -1 \\ 2 & -1 & -2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

This matrix is guaranteed to be nonsingular by the center manifold theorem, and indeed we find that is the case. The solution is $\alpha = \gamma = -1$ and $\beta = 0$, so $y = -x_2^2 - x_2^2 + \cdots$. Substituting this back into the original equations for (x_1, x_2) gives the dynamics on the center manifold:

$$\begin{aligned} \dot{x}_1 &= -x_2 - x_1^3 - x_1 x_2^2, \\ \dot{x}_2 &= x_1 - x_2 x_1^2 - x_2^3, \end{aligned} \tag{5.41}$$

up to terms of cubic order. The dynamics of (5.41) is nontrivial, and to study it we must use some additional tricks—we will develop these in the next chapter (see §6.3). We will find that (5.41) has the dynamics of a spiral focus. This implies, according to Theorem 5.9, that the origin of (5.40) is asymptotically stable. ∎

5.7 Exercises

1. Find all the homoclinic and heteroclinic orbits for the Hamiltonian

$$H(x, y) = \frac{1}{2}(y^2 + x^2) - x^4.$$

What are the stable W^s and unstable W^u sets for each of the three equilibria?

2. Consider the system (5.6) with Hamiltonian (5.3).

   (a) Find the equilibria. You should verify that $p = (x^*, \tfrac{1}{2}ax^{*4})$ is an equilibrium when $x^* = 0$ or is a root of the polynomial $q(x) = -4 + 12ax + a^2x^6$. Show that when $a \neq 0$, q has exactly two real roots and hence that there are three equilibria.

   (b) Show that the origin is saddle. Find its eigenvalues and eigenvectors.

   (c) Set $a = 1$, and find the new equilibria numerically. Show that that one is a stable focus and the other an unstable focus.

   (d) Investigate, using phase plane software, the dynamics of this system. What are the stable and unstable sets of each equilibrium?

3. Like the Lorenz model (1.33), the Busse–Heikes model describes three spatial modes in a convecting fluid, but in this case the fluid is rotating (Toral, San Miguel, and Gallego 2000). In one limit the model becomes

$$\begin{aligned}\dot{x} &= x\,(1 - x - (1+\delta)y - (1-\delta)z)\,,\\ \dot{y} &= y\,(1 - y - (1+\delta)z - (1-\delta)x)\,,\\ \dot{z} &= z\,(1 - z - (1+\delta)x - (1-\delta)y)\,,\end{aligned} \tag{5.42}$$

   where $\delta > 0$, and (x, y, z) represent nonnegative mode amplitudes.

   (a) Find all the equilibria and characterize their stability types as a function of δ. (*Hint*: There are eight equilibria: the origin, three solutions with one nonzero amplitude, three solutions with two nonzero amplitudes, and one with all three nonzero.)

   (b) Show that the quantity $R = x + y + z$ obeys a simple self-contained equation and that if $R(0) \neq 0$, then $R(t) \to 1$ as $t \to \infty$.

   (c) Assume that $R = 1$ and reduce (5.42) to a set of two equations for (x, y). Show that these equations are Hamiltonian with $H = \delta xy(1 - x - y)$.

   (d) Give a complete discussion of the dynamics of this model in the positive octant.

4. Using the integral (5.13), find the unique bounded solution to the forced system

$$\begin{aligned}\dot{x} &= -x,\\ \dot{y} &= y + \sin(t)\end{aligned}$$

   for an initial condition $\sigma = (x_o, 0)^T \in E^s$.

5. Show that any bounded fixed point $x \in C^0(\mathbb{R}^+, U)$ of the operator T defined by (5.17) is a C^1 solution of the differential equation (5.10). (*Hint*: Differentiate $x = T(x)$ with respect to t, remembering to differentiate with respect to all the places that t enters on the right-hand side.)

6. Show that if $L : X \to X$ is a linear operator on a Banach space, and $\|L\| \leq c < 1$, then the operator $I - L$ is invertible. (*Hint*: Consider the formal series expansion $(I - L)^{-1} = \sum_{k=0}^{\infty} L^k$.)

7. Here, you will show that the stable manifold theorem implies an equivalent unstable manifold theorem.

   (a) First, let $\hat{x}(\tau) = x(-\tau)$ in (5.10) and obtain the ODE for $\hat{x}$. This will give an equation similar to (5.10) but with $A \to -A$. Now, show that stable manifold theorem for the new equation implies the existence of a Lipschitz graph W^u over E^u.

   (b) Transform back to $t = -\tau$, and obtain the explicit operator T equivalent to (5.17) for the unstable manifold. Take care to keep track of all the minus signs!

8. Consider the system on $\mathbb{R}^2$ given by

$$\begin{aligned}\dot{x} &= -x + y^2,\\ \dot{y} &= 2y + xy.\end{aligned}$$

   (a) Find E^s and E^u for the fixed point $(0, 0)$.

   (b) Construct successive approximations $(x_i(t), y_i(t))$, $i = 1, 2$, to the stable manifold $W^s(0, 0)$ by applying the operator T, (5.17), to the initial guess $(x_o(t), y_o(t)) = (0, 0)$.

   (c) Compare the approximations in (b) with a power series expansions for the stable and unstable manifolds using the techniques of §5.6.

   (d) Using your favorite software, plot the functions you constructed and some numerical solutions of the differential equations. Compare the manifolds that you compute with the solutions.

9. Consider the system

$$\begin{aligned}\dot{x} &= x^3 - 2xy,\\ \dot{y} &= -y + x^2.\end{aligned}$$

   (a) Find the first few terms in the power series expansion for the stable and center manifolds of the origin.

   (b) Study the reduced dynamics on the center manifold. Classify the equilibrium.

   (c) Compare your analytical expression with numerical orbits generated by your favorite software package.

10. The three-dimensional system

$$\begin{aligned}\dot{x} &= y + 2z + (x + z)^2 + xy - y^2,\\ \dot{y} &= (x + z)^2,\\ \dot{z} &= -2z - (x + z)^2 + y^2\end{aligned} \tag{5.43}$$

    has a nonhyperbolic equilibrium at the origin.

(a) Find a linear transformation to write (5.43) in the form (5.34).

(b) Find the quadratic approximation for $W^c(0, 0, 0)$.

(c) Obtain the reduced dynamics (5.36) on W^c and use your favorite software package to study it. Is the origin stable or unstable?

11. Consider your adopted system of quadratic differential equations (recall §1.7 and Exercise 1.10) for the chaotic values of the reduced parameters. Use the techniques of this chapter to study the stable, unstable, and center manifolds of one of the equilibria.