intelligent 3D blackboards for geometric research

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1 Context

The semantics of computer graphics descriptive systems and programming languages can now begin to support entities and operations meaningful to domains such as theater and differential geometry. In the domain of performance and theater, one can begin to design high-level "director languages" to define the motions or behaviors of interacting 3D sprites. Such languages build on models of appearance and behavior abstracted from the highly evolved history of human performance (choreography, staging, mime, etc.). Highly evolved models of manifolds and more general geometric and topological structures also exist in the domain of mathematics and theoretical physics, but to date, interactive 3D visualization systems have not found wide use in the mathematical or theoretical physics communities. Within the field of differential geometry, perhaps the best known visualization system is the Geometry Center's *geomview* ([3], [6]), which has been coupled to a variety of numerical research applications, and to general algebraic systems. What *geomview* lacks in graphics sophistication is offset by its accommodation of structures and operators meaningful to working geometers. The next step is to couple such systems to 2D and 3D direct manipulation environments.

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2 an approach toward a geometric workspace

We seek 3D interfaces, or for that matter, any geometric workspace at all which approaches the suggestiveness of freehand chalkboard but couples to algebraic subsystems. Broadly, there are two ways to enrich 3D interfaces to make them more useful for geometers and topologists: by making "idiot-savant" manipulables with just enough physics to permit topological operations such as those described in section (3.3.2), and by linking manipulables to computational engines such as symbolic algebra systems, numerical analysis systems, and knowledge representation databases tuned to geometric information.

The general goal is not verisimilitude, or a mathematical equivalent of photorealism, which is generally not a strategic objective, but flexibility, expressiveness, and easy definition in terms familiar to geometers, topologists or allied researchers, as the following examples may illustrate.

One of my goals is the synthesis of symbolic algebra languages and numerical analysis tools into 3D interfaces, with converse feedback from 3D manipulation to algebraic representations of geometric structures.

3 applications

3.1 optical geometry

3.1.1 visualizing optics in *non-flat* spacetimes

What are the dynamics of test particles near a singularity? What would one see in the neighborhood of a singularity? This general relativistic grandchild of one of Einstein's early questions is distinct from the problem of drawing geodesics or lightcones in a (general) spacetime. The physics and phenomenology of optical geometry have been worked out since 1986 [1] so it should be possible to immerse students and researchers inside relativistic virtual spaces in order to design experiments such as those which would be used to test recent work (at Stanford) by Robert Wagoner and a student on shock oscillations near black holes.

3.1.2 properties of geodesics in spaces of (variable) negative or positive curvature

William Thurston and colleagues have produced visualizations of what one would see inside hyperbolic 3-space, a metric of constant negative curvature. (See the video NotKnot [2].) Building 3D interfaces for optics in space-forms could serve as a warmup for the previous challenge.

3.2 quantum cosmology

3.2.1 visualizing evolution of spacetimes in dimensions 1-4

In addition to the visualization of geodesics and optical geometry of classical relativity, inflationary cosmologies present additional challenges. Andre Linde's superposition of both classical quantum and stochastic processes [4] generate highly non-uniform geometries and even arbitrarily complex topological types. [1]

3.3 topology

3.3.1 *nonmetric* properties of shapes

Often, the particular graphical properties of a rendered object, such as texture, intrinsic color, reflectance, are much less important to the mathematical researcher or physicist than certain general geometric or topological attributes. One example comes from the isoperimetric inequality which is typically illustrated by drawing a pinched 3 manifold M shaped like a pinched, two-holed torus. One would like to knead such a solid, make various codimension-1 slices Σ, so that

$$M \backslash \Sigma = M_0 \cup M_1 \tag{1}$$

and read off the volumes to compute, for example, the ratio

$$\tau[\Sigma] \equiv \frac{|\Sigma|}{\min(|M_0|, |M_1|)}. \tag{2}$$

[1] The evolution equation for the metric is reminiscent of an inverse heat equation, which leads to exponential blowups in the super-universe. *q.v.* Linde

as a function of Σ. (If $dim(A) = n$, $|A|$ represents its n-volume.) $\inf_\Sigma \tau[\Sigma]$ plays a central role in the isoperimetric inequality and analysis on manifolds.

3.3.2 visualizing homotopies, knot moves, and related operations in topology

Many topological arguments, for example in the classification of 3 manifolds, depend on some dynamic operations ranging from crossing links in knots to excision of n-cells, glueing in simplicial complexes and continuous deformations – homotopies. A set of virtual "ropes" and "patches" which could support at least these standard moves in topology of knots and 3 manifolds would be extremely useful and a boon to students of topology. A spectacular example of such dynamic topology is the Geometry Center's video of an eversion of a 2-sphere, an illustration of a theorem by Stephen Smale. However, one could appreciate the eversion much more readily if one could interactively drop tracer dyes on one side of selected oriented patches of the sphere as it distorts.

3.4 riemannian geometry

Quite a few problems in geometry and physics concern global integrals over manifolds of quantities such as area, scalar curvature, Ricci curvature, and other norms of the curvature form and fundamental forms. To date, there has been little point in illustrating such problems using 3D visualization systems, but the theory has evolved to the point where 3D systems may prove useful.

3.4.1 variations by total curvature

Yet another global geometric quantity, and arguably the most natural "physical" quantity after total area to study, is the square norm of the mean curvature. Let ϕ be an immersion of a surface M into R^3, $\mathbf{H}$ the mean curvature vector on ϕ[M], *1 the area 2-form induced by ϕ, and define the total mean curvature functional W by

$$W[\phi] = \int_M |\mathbf{H}|^2 * 1 \tag{3}$$

The functional W is conjectured to be minimized among all surfaces of genus > 0 by the stereographic projection of the Clifford torus in S^3. This simply stated problem has generated a rich field of work suitable for 3 and 4-dimensional interfaces. (see below)

3.4.2 problems associated with visualizing structures in dimensions 4,6,7,8 which are critical dimensions in much deep, exciting theory in geometry and physics

One of the most basic problems with most 2D and 3D graphics systems is that they are designed to faithfully render structures of dimension less than or equal to 3. Some interesting work with translucent projections [5] have attempted to render 4D structures, but the problem remains that it is rather cumbersome from the point of view of a "naive" mathematician to define a custom immersion of an n-D manifold into a 3-dimensional space, even in cases where there exist highly-developed mathematical idioms describing some standard structures, such as covering spaces, vector bundles, or even affine projections of Hausdorff rectifiable sets. Such idioms can serve as interfaces for indirect manipulation of higher-dimensional geometry.

4 author's work

The common needs that thread these applications together include a description of graphical structures which can encode algebraic structures comprehensible to geometers and topologists, a set of 3D renderings and manipulation primitives which express mappings (gestures) characteristic of differential geometry and topology, and a language to match, with syntax close to the syntax of working mathematicians or physicists.

My own interest bifurcates into two categories: (1) the study of variational problems arising from total (mean) curvature, which lie in the intersection of riemannian geometry, geometric measure theory, nonlinear partial differential equations, and global analysis; (2) the visualization and manipulation of geometric and topological structures.

I've created simulations which transform a set of nonlinear pde's in Mathematica to a surface rendered via a variety of systems, including MathView, *geomview* and Renderman. The pde's include variational equations for flow by gradient of area, curve flow induced by mean curvature on a surface of revolution, perturbation by total mean curvature, and Linde's superposition of stochastic and quantum flow of a scalar field on 2-D spacetimes.

Some more particulars: in the case of the Willmore problem [7], the flow is by the gradient of an integral quantity defined on a compact surface M in a fixed ambient space, (R^3 or S^3). The variational problem is complicated by the fact that the functional is invariant under conformal maps of the ambient space. It would be convenient to have a set of operators, with which one could easily define and apply nonlinear geometric transforms to ambient spaces or submanifolds. In a prototype first presented at MSRI's Visualizing Geometric and Topological Structures workshop [5], I tried to demonstrate how useful it was to "drive" a 3D visualization system (*geomview*) directly from a symbolic algebra system (*Mathematica*). The setup, however, left much to be desired. For example, it would be quite useful to be able to run a glove over surfaces generated by the evolution scheme to seek umbilic points because such points ought to remain umbilic under conformal maps. Although one could always write special-purpose functions to detect such points for these particular surfaces, the moral is that we need 3D presentations which carry enough algebraic and differentiable structure to make low-level coding unnecessary. Furthermore, it would be quite useful to define compactly supported functions directly on the manipulable. What's important is not the precise parametrization of the support domain but the fact that it is compact, and that it contains or excludes certain geometrically interesting features.

References

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