A Laboratory for Geometric Performance [1]

Sha Xin Wei June 1998

I. Introduction: Creating Geometry

Why is it that when two students of differential geometry work together, it's more natural to turn to a blackboard than to a keyboard? How do we inscribe and work out mathematics with marks on paper or on a blackboard, and how is this different from typing math in TeX? Is mathematical creativity mediated by freehand writing and sketching in ways that are not captured by traditional keyboard-and-mouse writing systems? How do computation or automated symbol manipulation augment or degrade a writing system for mathematical work? What forms of writing, computation and symbolic encoding help or hinder the practices of inventing new mathematical structures, making new conjectures, or convincing oneself of the truth of a claim?

I would like to consider these questions in the context of a specific subdomain of geometry. This is a proposal to construct a formalism and an experimental computational environment that supports as well as possible a fragment of differential geometers' practice. I would like to analyze in detail what geometrical work can or cannot be performed easily in a hybrid medium provided by a generalized writing system (a so-called multi-modal geometrical liveboard) spanning freehand sketches, mathematical text, symbolic and numeric computation, and manipulable diagrams or graphics. The critical part of this project will be informed by insights from literary and performance studies as well as the mathematical sciences.

Practically, this work benefits students of differential geometry or topology, people who need to work with nonlinear or multi-dimensional information, and artists and designers who wish to shape computational material using freehand sketches. A technical payoff of this project should be a richer set of representations of topological and geometric structures that provide good grips for the construction of a subsequent generation of computational media.

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1 *This is a preliminary proposal for a research project, part of which may constitute a dissertation.* *I thank experts in central and allied disciplines for advice, criticism and encouragement: Niklas Damiris, John Etchemendy, Stefano Franchi, Ron Karidi, Larry Leifer, Tim Lenoir, Rafe Mazzeo, Robert Osserman, Richard Palais, John Perry, Alice Rayner, Ben Robinson, Brian Rotman, Warren Sack, Brian Smith, Rick Sommer, Pat Suppes, Barbara Tversky, Tom Wasow, Brian White, Helga Wild, Terry Winograd.*

The two major theses of this project are:

I. *Non-textually mediated forms of abstraction are employed by geometers, and can be sustained in a hybrid graphical, algebraic, numerical writing system*. There are some language-like features of graphical, algebraic or numerical presentations on which humans capitalize when creating and reasoning about geometry. Among the most important of language-like practices is the ability to make abstractions from the signs presented in two or three-dimensional form, and the ability to use these graphical signs to perform mathematical operations with geometric entities that are more "abstract," and not directly visualizable. These abstractions from the graphical signs allow geometers to work with continuous entities (such as riemannian manifolds), infinite entities (such as function spaces, or minima defined by a limiting process), and inequalities or comparisons (such as comparisons between a model space and non positively curved metric spaces).

II. *A fruitful and efficient way to support a smoothly hybridized computational geometric writing system is to use high-level encodings of differential geometric structures*. Geometers in the wild move smoothly between multiple modes of presentation, and I wish to sustain such practice in an integrated computationally augmented medium. I claim that the presentation and manipulation of geometry are better mediated by high level structures close to the forms common in colloquial mathematical writing and sketching, than by the lower-level structures common to computer graphics and computational geometry. I.e.. instead of polygons and related primitive structures with associate algorithms in traditional procedural languages, I claim that richer multimodal, graphical manipulation systems can be constructed based on representations of differential geometric entities such as riemannian manifolds and function spaces. This should eliminate some of the circumlocutions needed, for example, to encode arguments appealing to abstractions of visual intuition in "natural language" text; or to represent geometric operations in low level programming languages or data structures.

The major conceptual work of this project is to interpret the terms used in the two theses in an appropriate sense, and to justify those interpretations. This first section lays out the theoretical and experimental aspects of the project. Section II describes how this project is interesting from philosophical, mathematical and computer science perspectives. Section III gives a brief plan of work. I provide a few examples in Section IV, and in accompanying animations. Section V (Appendix) gives some references and a survey of related work.

FIGURE 1. A rudimentary example of a classical geometrical writing and reasoning system.

Theoretical Aspects of the Project

I would like to study the interplay between the construction of geometric structures and the constraints of computational media technology, an analysis of mathematical reasoning embodied in mathematical "writing."[2] To perform a contemporary study of problems related to the representation, invention and expression of mathematics, one should study not only the logic of mathematical knowledge but also how the form and legibility of symbolic media -- particularly computational media -- shape mathematical practice and mathematical discourse. (For a more extended discussion on writing, performance and modes of mathematical practice, see [Sha 1998b].) By mathematical practice I refer to what humans do as they create and transform algebraic, geometric or analytic structures, whether they be novices or experts, clients of mathematics, or professional mathematicians.[3]

2 There is a rich literature on the relation between mathematical knowledge and mathematicians' physical and social experience: writings, utterances, and the like. Classical sources include Husserl's phenomenological studies of mathematical experience (eg. *Origins of Geometry* in Crisis of European Sciences) and Lakatos' study of the discourse of mathematical discovery (Proofs and Refutations). A contemporary interest in body-based and more narrowly, neural-based, experience and consciousness is represented by Varela, Thompson and Rosch's The Embodied Mind. Connes & Changeux explore issues directly related to embodied mathematical process in their popular Conversations on Mind, Matter and Mathematics.

3 One can view this project naturally in the context of what Barwise and Etchemendy call *heterogenous reasoning* -- logical inference using multiple modes of representation. In [Allwein & Barwise 1996] they lay out the arguments for a study of visual representations as a basis for *non*-linguistic reasoning. Another context is the study of the phenomenology of the *practice* of mathematics, expressed in the tension between geometrical "reasoning" and "writing." Gian-Carlo Rota's Indiscrete Thoughts contains thoughtful essays about the phenomenology of mathematics. For a parallel discussion, see Brian Rotman's functional semiotics of mathematical discourse in Ad

This study will serve dual purposes. The first is to augment certain modes, perhaps create new modes, of mathematical practice. And the second is to understand writing as performance rather than information recording, with a particular focus on the creation and manipulation of “continuous” structures. At minimum, this study should produce a computational system -- a geometer’s workbench (GW) -- in which we can test some hypotheses about how humans invent, play with, and reason about geometrical and topological structures. I will sketch an architecture for such a laboratory later in this section. First, let's start with some discussion of the represention of geometrical structures and the actions that geometers perform with them.

We can distinguish at least four modes of representation:

• Discursive (assertional) representation. For example, we can write a statement like “An Einstein metric is a critical point of the scalar curvature integral” in "natural" language.[4]

• Algebraic or symbolic representation. For example, we can describe a metric as a symmetric two-form, using some encoding such as $g(x,y) : T_pM \times T_pM \to R$.

• Parametrized or numerical representation. For example, we can compute metric connection components or solve the differential equations for a Jacobi field in terms of the explicit local coordinates for a surface.

• Graphical representation. Typically, a visualization that's "isomorphic" to the structure in question will not exist, but an interesting issue is how experienced mathematicians generalize correctly from reduced and abstracted sketches of geometric structures.[5] Often, a sketch merely functions as a mnemonic in a discursive representation, and may not encode anything like a structure or a proof strategy in a form amenable to algorithms of computer algebra. But sketches can play crucial roles in the construction of a proof. We use graphical representations in roles ranging from those that iconically aid the memory, and those that illustrate a simple case, to those that explain a proof, and those that can be formally (mechanically) mapped to a (part of a) logical proof.

Let’s consider a toy example in order to clarify what these modes of representation could be:
Say that we wish to study geodesics -- length-minimizing curves -- on standard and non-standard tori.
A *discursive-symblic representation* would be written like this declaration:

```
Let torus1 be a standard torus:
torus1 = torus[ Sqrt[2],1];
```

An example of an *algebraic representation* would be a procedure to define a geodesic as the solution to a differential equation, here packaged as a function named `geodesic[]`. Note that in this special case, the symbolic algebra engine can produce an exact solution to the ODE.

```
g1 = geodesic[torus1,{Pi/2,0},{0,1},15];
```

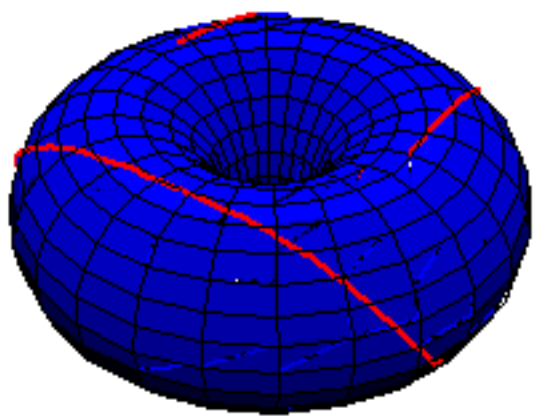
Infinitum.

4 In this context, "natural language" refers to the expressions that would be used in conversation by human expert and student mathematicians.

5 *ND: Need to discuss difference between shape & structure as predicate on geometry or algebra.*

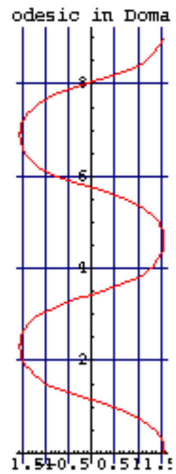
```
Calculating Christoffel symbols...
                      -2 (Sqrt[2] + Cos[u]) Sin[u]
       {{{0, 0}, {0, ------------------------------}},
                     5 + 4 Sqrt[2] Cos[u] + Cos[2 u]

               -2 (Sqrt[2] + Cos[u]) Sin[u]
         {{0, ------------------------------}, {(Sqrt[2] + Cos[u]) Sin[u], 0}}}
              5 + 4 Sqrt[2] Cos[u] + Cos[2 u]

       The differential equations for the geodesic are:
                                                     2
       (Sqrt[2] + Cos[u[s]]) Sin[u[s]] v'[s]  + u''[s] == 0
       -4 (Sqrt[2] + Cos[u[s]]) Sin[u[s]] u'[s] v'[s]
       ---------------------------------------------- + v''[s] == 0
           5 + 4 Sqrt[2] Cos[u[s]] + Cos[2 u[s]]
       with the initial conditions:
                                  Pi
       u'[0]= 0, v'[0]= 1, u[0]= --, v[0]= 0
                                  2
       Solving differential equations using NDSolve...
       Solved differential equations.
       Preparing plot of geodesic in domain...
       Preparing plot of geodesic in surface...
```

From this “algebraic” description we obtain a *graphical representation*:



which in the ideal setting would be directly manipulable via gestures.

Of course, in order to produce such graphics, the algebraic representations will be coverted at some stage to numerical representations.   Here the numerical solution to the geodesic ODE is plotted in a parametrization domain of the torus.



Ideally, manipulating ("re-writing") elements in one representation should modify all the other representations. For sufficiently focussed domains, useful models may be found. In any case, it must be as easy as possible to move back and forth between modes of representation.

While one can now automatically generate a manipulable graphic representation from a symbolic representation of a geometric object as in the example of finding geodesics on torus, the reverse is quite difficult, even for simple geometric objects. That is, it is difficult or impossible in most current general-purpose visualization systems (eg. AVS, SGI Explorer, Geomview [Gunn1]) to modify an algebraic or symbolic structure via direct manipulation of a graphical representative.[6] For example, it's easy to compute and plot a surface as a graph of some function f, but difficult to select by gesture a sub-domain U of the surface in a graphical interface and perform some geometric analysis with its pre-image $f^{-1}(U)$. Why is this the case? Setting aside computer graphics, one fundamental obstruction is the lack of mathematically sophisticated yet efficient formal representations, encodings, of geometric objects in a model shared by graphical and symbolic computation systems. (I'll indicate what I mean by these systems in the description of the GW's architecture.) *A principal technical goal of this project is to define such a set of representations.*[7]

One of the strengths of a writing system that spans multiple modalities is that we can supplement one representation by information from another. For example, computing global information about a manifold, such as its genus or closed geodesics may be algebraically and numerically infeasible, in which case we turn to the discursive representation for help in drawing conclusions. As another example, solving the ordinary differential equations for a surface may simplified if we can prove some

6 Geometer's Sketchpad does let one modify an algebraic-symbolic representation via gestures, but only in a restricted domain of classical plane geometry.

7 For notes on a formalization of such representations and their relations, see [Sha 1998a].

fact about the symmetries of that surface, then have the system automatically use this fact in its algebraic or numerical computation. So we come to the following key questions: What's an adequate way to represent continuous and other non-discrete geometrical structures? What's an adequate way for humans to manipulate such geometrical entities in a multi-modal writing system?

Although the problems of adequately representing mathematical structures and theorems are interesting in their own right, this project aims to accommodate not only descriptions but *actions* as well. Indeed, I believe that merely illustrating known facts or well-understood structures would not warrant an ambitious project. The GW is not to be just an illustration system (such systems exist) but a *computational writing* system, where writing is taken in a generalized, multi-modal sense -- discursive, algebraic-logical, numeric, graphic. What is an adequate form for the fragment of geometers' research practice, of writing and discourse, that can be computably represented? The formalism should sustain not only logical assertions, but also constructions, calculus of variations, and function estimates (i.e. inequalities).[8]

One aim here is to see whether novel semiotic elements may be needed to support standard practices of geometers and analysts, and if so, what they might be. Representing mathematical action poses different problems from representing mathematical knowledge. My approach, like that of many mathematicians (see eg., S. Feferman, J. Christy, K. Devlin), leaves mathematical understanding along with agency and interpretation in the human.[9] Instead of a project to formalize or automate

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8 This is the old distinction between "knowledge-that" and "knowledge-how" -- heuristics.

9 Feferman expresses what I believe is the Reasonable Mathematician's opinion on this issue:

> [I]t would be ridiculous to think that anything like such a search through proofs takes place in the activity of working mathematicians. How it is that they actually arrive at proof is through a marvelous combination of heuristic reasoning, insight and inspiration (building, of course, on prior knowledge and experience) for which there are no general rules, though some patterns have been discerned by Polya and others: there is no formula for mathematical success. It is only when one finally arrives at a proof that one can check (mechanically, in principle, but not in practice) that it does indeed establish the theorem in question.

...

> So on the face of it, mathematical thought as it is actually produced is not mechanical; I agree with Penrose that in this respect, *understanding* is essential, and it is just this aspect of actual mathematical thought that machines cannot share with us. [*review of Penrose's* <u>*Shadows of the Mind*</u>, 1994 italics original]

In a more radical analysis, Rotman proposes a functional semiotics of mathematics which is centered on the embodied processes of the construction of signs and of interpretation. He predicates his analysis on locating agency not in disembodied symbols but in some ratiocinating being. To these two positions on embodied cognition, we contrast Newell and Simon's Physical Symbol Systems Hypothesis which formalizes the attitude taken by classical artificial intelligence researchers who would create "automated reasoning systems."

understanding, this is a project to study how the emerging hybrid writing systems (as defined above) augment and constrain the construction of geometric phenomena and statements about them. As such it is a companion, or a prologue, to any project on formally modeling mathematical reasoning. Since the general problem is difficult, it makes sense to ground the study in some well-established, rich domain of mathematical theory and practice.

Differential geometry, along with geometric measure theory, geometric analysis and their applications is a good candidate domain for this study. Like mathematical practice in general, there tends to be less ambiguity in the language of differential geometry than in everyday ordinary language; that is differential geometers are more likely to agree on the use-meaning of a geometrical sign than users of more general language. Many central problems and intuitions of differential geometry are intertwined with physics, and as such it is a reflection upon how humans come by their geometric intuitions. Perhaps the most important reason why differential geometry could supply good applications for a GW is the following: Constituting theories about extension, approximation, variation, and curvature, but abstracted beyond merely a theory of the visual, differential geometry provides a strong "boundary object" [Leigh Star] -- a source of counter-examples -- to mimetic representations and simple mimetic theories of geometric cognition. A glance at contemporary differential geometric research reveals that the geometrical is not simply a description of the visual. It concerns essentially non-discrete structures and uses non-constructive arguments, which challenge conventional logical models and computer representations. Many deep geometric intuitions bear no mimetic resemblance to shapes or processes in our nominally Euclidean perceptual space. And yet, geometers and topologists do use graphical representations to work out proofs. And experts make sketches that capture the essence of true generalizations. For example, certain arguments of classical general relativity are performed with spacetime diagrams. (FIGURE 2) Many topological and geometrical results in this genre depend crucially on arguments that are carried out “in” the diagram.

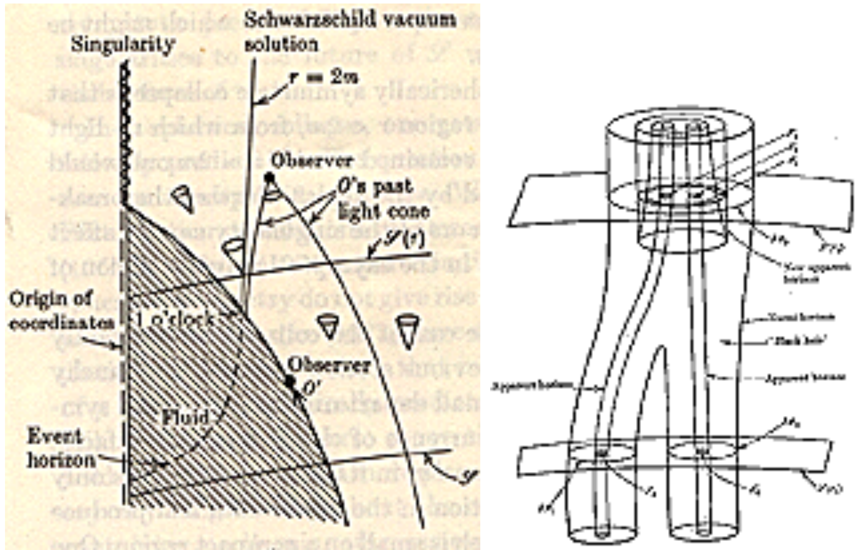

FIGURE 2. Hawking and Ellis: Finkelstein spacetime diagram of gravitational collapse (p. 309) and coalescence of two black holes (p 322). Relativists using these geometrical techniques argue proofs though such diagrams.

A more modern example is the video of the eversion of a sphere which explains key elements of Smale's proof of the sphere eversion theorem. (*Inside Out Video* from A.K.Peters) In both cases, the graphical representations are clearly not mimetic of visual phenomena from everyday life. Moreover, the representations can be precisely described in other modes as well -- in discursive form, and in some instances in symbolic (eg. equational) form or even as numerical simulations.

Of course, not all geometric structures and arguments are equally faithfully encoded in all of these modes. Diagrams can be merely tokens that syntactically function rather like an identifier or a Chinese character.[10] I claim that a multi-modal writing system's usability will depend to a large extent on how smoothly the human can shift between writing and reasoning in one mode to another. But the responsibility for correct heuristics -- what operation do I perform next? -- rests on the human, not on some inference engine.

Since a central aim of this project is to design and build models for a fragment of geometry and

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10 For example, to the expert geometer, $\begin{array}{c} \mathbf{E} \\ \big| \, \pi \\ \mathbf{M} \end{array}$ is a token denoting "a fibre bundle E over a base manifold M." This token bears a graphical residue of its origin as a "working diagram" of such geometric structures.

analysis sufficiently rich to support “really-existing” mathematical discourse, the success of these models will be measured in part by their usability by experts and novices in the chosen domain. We can measure the success of such a project by seeing (1) to what extent experts and novices find useful ways of creating or learning about geometrical theories, and (2) what new insights we can learn about geometrical reasoning.

The specific application will be chosen based on mathematical interest, appropriateness and tractability for this study. Possible specializations include: surfaces of finite total curvature; currents a la Fleming, White; geometry of Lie groups; harmonic maps of surfaces. Let me propose a few scenarios which, if they can be supported, could be sufficient tests of a GW. They should illuminate some of the meta-mathematical issues that this project aims to understand.

First scenario: Define a surface as the boundary of a geodesic ball in an abstractly defined manifold (eg, a Lie group). Specialize to case in which we can look at a cross section with a R^2. Vary the metric algebraically, and move the center around in the cross-section by a gesture. Inspect the surface.

The second scenario (FIGURE 3): Define --in mathematical English -- a continuous family $\{\Sigma_t\}$ of complete hypersurfaces as level-sets of a smooth function f: M -> R, where M is a Riemannian manifold of unspecified dimension and metric; $\Sigma_t = f^{-1}(t)$. Specialize to a particular situation, say $M = R^3$, and $f(x) = |x_1|^{2} + |x_2|^{2} - |x_3|^{2}$, and call for a graphic picture over a range $1/2 \leq t \leq 2$. Then by a gesture specify a compact domain in M meeting $f^{-1}(1) = \Sigma_1$, and declare a normal vector field, parametrized by a freehand plot. Say: perturb the surface Σ_1 by this vector field. Then evaluate an integral quantity over the perturbed patch. In order to be more than "mere hand-waving," the gestures must deterministically modify algebraic or discursive representations of the geometry.

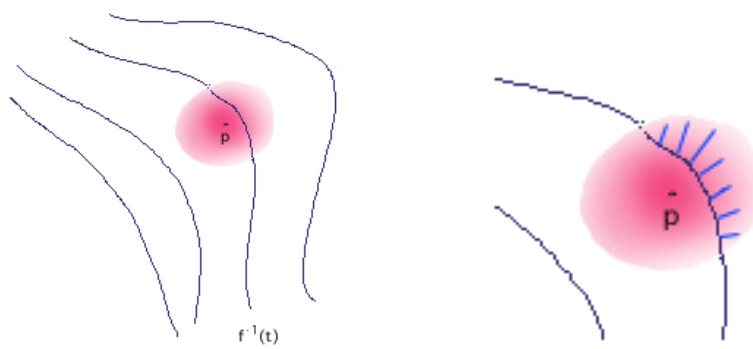

FIGURE 3. Define a foliation by level sets of f(x). Define compact set (shaded) by gesture. Perturb a leaf by a compactly-supported vector field, defined by gesture.

It should be clear that I do not want to build "tools" special to these situations, but a more general system in which a mathematician can easily construct for himself or herself special structures and operations in multiple modes -- discursive, algebraic, graphical, etc. To support these applications, geometric models need to be constructed in multiple types of representation.[11] For example, a geodesic ball will be representable in an invariant, coordinate-free form, *or* in a specific dimension and coordinate system, *or* as a set of numerical solutions to a system of ODE's, *or* in a graphical form (eg. via an immersion or via pullback under the coordinate functions to a domain in R^3 or R^2). The models need also some descriptions of more elementary notions, so one challenge will be to identify a minimal hierarchy of descriptions that will suffice. The basic models should include elementary set theory, point set topology, basic Riemannian geometry, and will draw upon existing work.

More specifically, here's a rough sketch of some relevant notions, to give a flavor of what is needed. It's important to note that symbolic systems already exist in the literature for certain specialized domains, though it is not easy to transfer results from one system or representation[12] to another. Keep in mind that we do not require or expect automated inference in these domains, though partial systems would be welcome.

• Point-set topology: cover, compactness, continuity, map, domain, range, inverse, homeomorphism, homotopy, limit;

• Measure theory: Lebesgue, Hausdorff measure, density, integral;

• Analysis: smoothness and analyticity of map, diffeomorphism; approximation; function spaces, inequalities from analysis;

• Algebra: certain algebraic structures associated to the geometric model, such as differential forms, the tensor algebra (J. Lee's *Ricci*) along with basic symmetries (Bianchi I,II, etc.); Lie algebra, if necessary (eg. G. Baumann's *Baecklund* and *Lie*).

• Geometry: manifold, atlas, metric, exponential map, (tangent, normal) bundles; area, curvatures, variations of geometric quantities, geodesic, Jacobi field, flows (geodesic, mean curvature, etc.)[13]

It remains to be seen what is the minimal model we will need to encode in order to do geometry in the chosen specialization.[14]

Experimental Aspects of the Project

The experimental part of the study will be the construction of a computational writing system or liveboard. But this research is concerned with geometric models and issues of representation and

11 ND: *Need to distinguish between sense modalities and modes of presentation.*

12 *ND: Need to make consistent use of terms like semantic system, symbolic system, representation, presentation, forms, structure, notation.*

13 Geometric measure theory provides another useful set of notions (polyhedral chains, mass and flat norms, currents, approximation). We'll defer a description of the encodings for these notions.

14 *ND: The deep problem of how come these rich math structures correspond to physical phenomena is the inverse of how the set of "writing representations" in XW's sense emerge genetically form a continuous field-medium.*

discourse practice, not with the engineering of software systems. Thus I intend to assemble a software framework using existing software systems to the greatest degree possible, in which I can instantiate and manipulate the representations. This environment would (1) allow direct manipulation of graphical structures, (2) provide notation and functions for algebraic (pattern-based) as well as numerical computation, (3) tie into an existing body of mathematical/computational literature -- mathematical notebooks, packages and texts. A key feature of this environment is that the geometric structures can be associated with non-visual, but encodable differential or algebraic structures. This has the added advantage that the experimental apparatus -- the computational writing space -- can be widely replicated.

At the coarsest scale, the architecture could have a tetrahedral structure comprising a direct manipulation environment, a symbolic algebra engine, a numerics packages, and a knowledge secretary. (See FIGURE 4.)

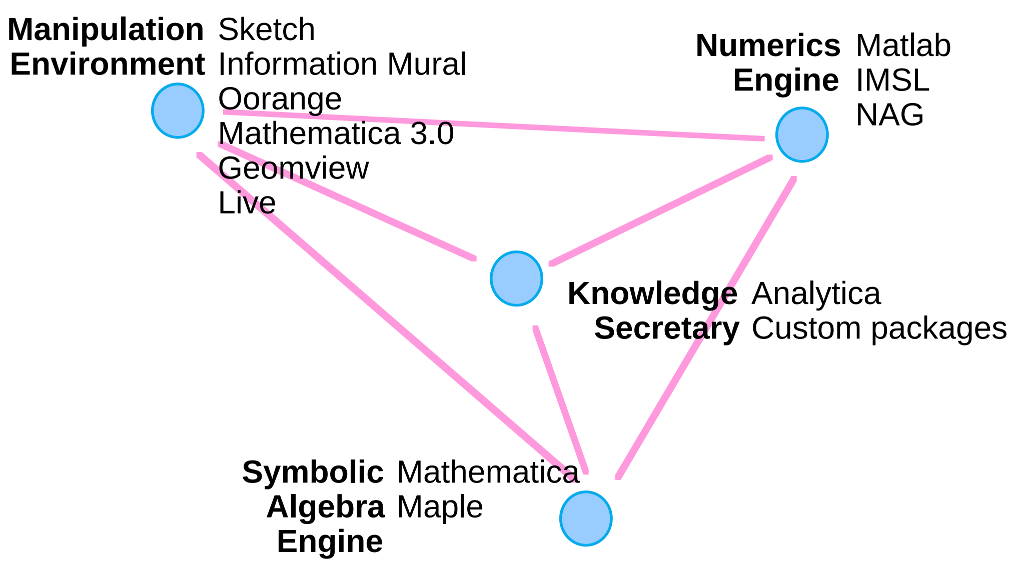

FIGURE 4. Architecture of geometric workspace -- component categories in bold, and possible instances in light face. (See the Proposal for an Intelligent Geometric Blackboard 1994)

The manipulation environment will be a hybrid between a live, structured 2D expression (TeX-like) editor and a 3D graphical manipulation system, likely to be some combination of Live3D, Geomview, the Mathematica 3 Notebook Front End, or C. Gunn and U. Pinkall's Oorange. [Gunn et al.] The symbolic computation engine may be Mathematica 3 or Maple. Numerical computations can be performed in Matlab. The knowledge maintenance system can be some specialized system such as Analytica [Zhao & Clarke], or an extension of a general system such as Hyperproof.

Algebraic, numerical and visualization tools abound, so their creation will not be my main ambition, although I expect to contribute to the design of some next-generation tools. Fairly robust examples of these components already exist and will make the construction of such a laboratory a feasible task. For the 1992 Workshop at MSRI on Visualizing Geometric Structures, I constructed a prototype system using Geomview, Mathematica and MathView (an experimental 3D viewer), in which one could create polyhedral chains and transform them under sequences of deformations of the ambient 3-space. For example, this gave an easy way to play with, in a very concrete, graphic fashion, the action of a one-parameter subgroup of the group of Mobius transformations. This prototype GW required a Silicon Graphics computer at the time, but within a few years, equivalent software facilities should be available on commodity computers. Direct manipulation systems may be based on work from many sources (eg. CSLI, Computer Science Department, Center For Design Research, CMU, Xerox PARC, MIT, SGI).

In the course of building and evaluating that prototype it became clear that more flexible and powerful formalisms for describing and operating on geometric structures were needed. How best to represent and encode sufficiently general geometric structures will be an important part of the research project. One approach may be based on clearly distinguishing mathematically meaningful definitions of geometric structures from corresponding graphical encodings in 2D or 3D rendering systems.

It is important to underline that I am not proposing to build a theorem prover, nor is this a proposal to simulate mathematical reasoning even in a weak sense, although we might take advantage of existing knowledge maintenance systems.[15] Such work may be quite useful, if we forego the demand for a fully automatic inferencing mechanism.

II. The Interdisciplinary Interest

This study should be interesting from three perspectives, that of the philosopher, the mathematician and the computer scientist.

The philosopher's interest

Studying how expert and student mathematicians construct and communicate knowledge about geometry or topology should yield insights about other forms of nonverbal human experience, experience that is not mediated by natural, textual language. Here I would like to distinguish between creative reasoning and communicating, emphasizing the former. There are several ways to get into the head of a reasoning being: by interviewing subjects in controlled circumstances (psychology), by studying the marks and interpretations that he/she makes (literary studies), and by reasoning over the

15 Ken Haase (MIT), E. Clarke (CMU) and many others.

evidence provided by these and allied studies (philosophy).

In this project, we begin with the study of how expert and novice mathematicians perform geometrical work in multiple modes of representation. But this becomes interesting as an investigation in the philosophy of mathematics, a study of mathematical experience, the discovery or construction of mathematical knowledge in geometry and analysis. How is non-textually mediated cognition or communication based on geometry or topology (ie. schematics, diagrams, algebra) possible? More specifically, I am interested in the delicate interplay between graphically(visually)-derived intuitions and geometric reasoning, and their coupling via algebraic and computational models. The aim here is to come up with a theory that doesn't pre-suppose naively mimetic representation. I propose to study the phenomenological and logical problems raised using multi-modal writing systems to manipulate structures or processes that a priori may be non-compact, or non-denumerable. This reinforces the choice of differential geometry or topology rather than Euclidean or projective geometry as the object of study. Let me elaborate this point. To study conjectures about generalized writing and what it affords, it is useful to study a broader range of discourse practices, with a larger range of systems of registering those practices. Although they may be easier to evaluate, theories about graphics or geometrical intuition based on Euclidean geometry risk being incomplete or distorted by a restricted notion of geometry and an identification of the visual with the geometric. If we wish to study a relatively deep fragment of human discourse and reasoning not mediated by natural language text, we can take advantage of differential geometry's abundance of rich, "non-literary" discourse practices.[16] By relaxing the restriction to the primitive geometric objects (eg. oriented polygons and spheres) and operations (eg. affine maps) found in computer-aided diagramming systems and other computer graphics applications, we may arrive at a more powerful description of non-literary media and expression.

This project, then, can be seen to have a more general relevance in philosophy. This provides some insight into how we work with symbolic structures, and may provide even some experimentally grounded understanding of how, or whether, the mode of manipulation of symbolic structure constrains and shapes the concepts that are inscribed in those structures.[17]

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16 The project should include more precise characterizations of "reasoning" and "discourse," and more precise distinctions among "linguistic," "literary," "graphical" and "diagrammatic" modes of representation.

17 One relevant question is the relationship between representation and referent, which is denied by Husserl, according to André Orianne [introduction to Levinas' <u>Theory of Intuition in Husserl's Phenomenology</u>]. Peirce's notion of an indexical sign, may be an appropriate way to understand the GCL's multimodal manipulable representations.

The mathematician's interest

I believe there can be a carefully moderated path between an uncritical adoption of computer software and an equally naive restriction of computer usage to TeX and email. Both attitudes can arise from unreasonable notions of computation or an overly restricted notion of writing. Keith Devlin compares using modern software to driving automobiles, and argues that, just as driving a car is a fundamentally different activity from "walking faster," using complex computer systems like Matlab or Mathematica and Geomview requires new norms of use. But Devlin's suggestive automotive metaphor overlooks an essential feature of computational media: that they are reflexive, time-varying symbolic systems. The computer makes possible numerical simulations, algebraic models, and graphical and textual representations that can modify themselves over time. If all one desired were illustrations of mathematical objects or arguments, then "pre-fabricated" illustrations or video recordings would suffice.[18]

Another metaphor common among software designers and users alike is the notion of software as a "toolkit." But a toolkit with a fixed set of primitive objects and functions does not suit the mathematicians' style. Freely inventing new structures and operations is integral to mathematical practice. This is why I characterize the GW as a generalized writing domain rather than a set of tools. Currently, a typical hour in a mathematician's exploratory work is spent doing some thinking, a bit of writing or doodling or calculation, then some more thinking. This finely structured dance between reflection and gesture does not mesh well with large computational tools such as TeX, Matlab, and a C compiler.[19]

A *seamless* algebraic, geometric, numeric, graphical and logical system should provide a richer writing medium that yields new practices in mathematical research. The ease and fidelity of translating between representations in a computational environment is often undervalued in the evaluation of how well it augments exploratory, creative work. But with increasing sophistication comes the realization that model-translation issues lie among those crucial to the construction of a system that can serve as an exploratory environment rather than merely a system for illustrating or recording previously derived results. Moreover, much of the strength of a system with multiple modes of representations comes from making it possible for the human to fluidly switch between marks that can be manipulated by an "automatic" algorithm, and marks that can be interpreted only by humans. Examples are the description of a curve as the orbit under a group action, and the space of metrics on a two-torus modulo conformal equivalence. In each case, we may profitably use a schematic rather than

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[18] And such use of computer technology would hardly justify its cost.

[19] Much mathematical work is "purely mental," and thus does not need any external material medium at all. But to the extent that some mathematical work is mediated by material instruments (hand gestures, paper, chalk, computer), the question is what qualities of a computer medium are fundamentally different and useful compared to other material media.

textual narrative to represent this. But under some circumstances, it may be quite useful to compute analogously defined orbits by numerical means.

But, setting aside historical contingency, mathematicians face limits on the flexibility and usefulness of computational media which stem from several fundamental assumptions commonly found in computer systems, notably:

- a reliance on a fixed set of primitive primitives (the “Point2D, Point3D” problem);
- the computational assumption that sets and operations are finite;
- overly explicit representations;
- restriction of structures to finite-dimensional vector spaces or graphs.

I will choose some structures or theories that traditionally have no high-level computational representation, and see what limits their expressibility. Examples include complete (unbounded) surfaces, inequalities in analysis, limiting sequences of maps and gluing constructions that preserve specific geometric properties.

There are a few precedents and related work that I discuss in the Appendix.[20] For numeric computation, there is the general purpose matrix computation system -- Matlab, and special applications like K. Brakke's Evolver [Brakke] for curvature-dependent relaxations of surfaces, to name but two of many examples. IBM/NAG's AXIOM is a notably powerful algebraic system. Geometric and topological visualization systems include Banchoff's animations of curves and surfaces; Thurston and the Geometry Center's visualization of hyperbolic manifolds; Hoffman and GANG's minimal surface visualization system (VPL, MESH), Pinkall's Oorange system for computing and visualizing surfaces; Palais' surface plotting software. However, to date these have not been integrated in any coherent, flexible, multi-modal environment. One limitation is that most either depend on special properties of geometric structures (eg. minimal surfaces’ Weierstrass representation or quaternion algebra), or use only a low-level model of graphics objects (eg. polygons in an explicit coordinate system). This issue of low-level models leads us to the computer science interest in the GW.

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20 See Sha94 and WWW slides for list of related software. See the Appendix for an extended discussion of extant mathematical software and their limitations that the GWL addresses.

The computer scientist's interest

This study should yield new structures or high-level languages for dealing with geometric (not just graphical) objects at a layer of description that's convenient for humans as well as algorithms. I have no quibble with graphics data structures and algorithms which are designed for computational efficiency. Indeed, the GW will rely heavily on efficient algorithms for manipulating low-level representations. But one technical aim of this project is to fill a gap between, say, the category of oriented polygons and the category of chalk marks, or between TeX and manipulable 3D graphics. Given encodings of geometry in algebraic, logical and graphical domains, we can explore the problems of reasoning and translating between multiple representations, in, for example, the conceptual framework laid out by [Barker-Plummer and Greaves].

The first version of the GW will rely upon modifications of existing visualization/manipulation technology, though in a second generation, the "liveboard" interface should take advantage of contemporary gesture-recognition systems[21]. A practical consequence of this work should be the design of a next-generation writing system, where writing is broadened to include geometric sketches. Current gesture-recognition systems and 3D interfaces map user-gestures back to primitive structures like point in space-time, or oriented polyhedron, that are appropriate for machine representations of computer graphics, but are too low-level to be mathematically useful.

One technical challenge in interface design is the "inverse" problem of modifying algebraic/structural representations via graphical ("3D" or "2.5 D") interfaces. The GW version of this problem seems more tractable than the broader problem of computer vision (to isolate and recognize arbitrary objects in a photographic or video image)[22]. Supplementing the techniques of computational geometry and computer graphics, we can disambiguate graphical or gestural input using high level algebraic or logical models. For example, if one picks and drags a curve that is defined as an immersion in a surface, the dragging can be interpreted as a tangential perturbation and be constrained to the surface.

The practical goal, of course, is to design rich, seamless workspaces for mathematical work in more "real-life" situations. This seamlessness is valued by designers of human-computer interfaces, but is not feasible unless we restrict attention to practices where the scope and meaning are circumscribed sufficiently narrowly.

Ideas from differential geometry are beginning to find applications in computer graphics (see eg. Grimm & Hughes, Tomasi), but they are generally aimed at representing *visual* structures in Euclidean spacetime. Generalized geometric structures can achieve greater encoding, transmission and computational efficiencies by factoring some of the representational baggage into different models. For

21 Eg. Landay et al's AGATE (derived from GARNET) at CMU, among others.

22 See review by D. Mumford, AMS Bulletin, ____ 1996.

example, by providing a unified encoding of intrinsic geometry (eg. Point, as opposed to Point-In-A-Plane-Specified-By-This-Normal), a computational system can defer much storage and computation until the place and time that it is needed.

III. Schedule of work

It's common to demonstrate the power and utility of a new method by recovering known but interesting results, and then proceed to demonstrate new applications. The strategy will be to define a parsimonious model for a fragment of geometry that is rich enough for sub domain of mathematical practice in which hybrid techniques have been used fruitfully. We will study if and how the hybrid method yields the known results more handily. When we have constructed a system in which one can work in a novel way in a known sub-domain of geometry, we can then apply the GW to domains in which new results may be obtained.

One mathematical domain of recent interest is the study of the evolution of geometric quantities such as area or an integral of curvature, ranging from Sethian-Osher's simulations, Evan-Spruck's analytical justification, the geometric measure theory (Federer, Fleming, Almgren, Brakke) and PDE theory which provides existence and regularity of elliptic equations (Simon), and the recent extensions by White and Ilmanen. We will demonstrate how an investigation of these evolution problems can be investigated more easily in a coherent, hybrid computational environment.

A domain in which new experimental results may be useful is the study of constant mean curvature surfaces making use of explicit constructions (Lawson, Kapouleas), discrete constructions (Oberknapp, Grosse-Brauckmann, Polthier), global properties of embedded cmc surfaces (Korevaar, Kusner), and studies of the moduli space of complete cmc surfaces via analytic techniques (Mazzeo, Pollack, Pacard). Experimental techniques by Oberknapp and Grosse-Brauckmann may be transplanted from their original setting in the Grape system, to our more general framework.

Briefly, here are the main steps in this project.

• Survey literature on formal representations of mathematical objects and theories related to geometric manipulation and reasoning.

• Choose a mathematically significant problem, domain and characteristic structures or methods.

• Design symbolic models for geometric and graphic objects. Attach numerical simulations/graphical renderings as needed.

• Couple symbolic and numeric engines to interface via shared models

• Evaluate by testing with professionals and students, perhaps adjunct to a course on differential geometry.

I've completed a preliminary cycle of this work, and am now preparing the ground for the project.

IV. Examples

Here are some primitive examples, constructed to illustrate some of the distinctions and problems in the proposal. They are *not* meant to illustrate what geometry in a good GW should look like. (Other examples exist in the form of Live3D manipulables, QuickTime movies, and some structures encoded in Mathematica.)

Example 1, geodesics on torus

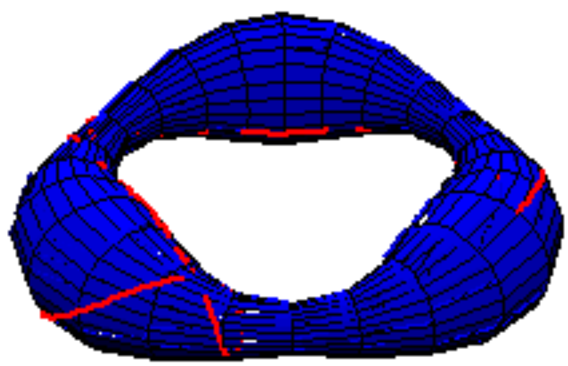
(See Body of manuscript.)

Example 2, lumpy torus

The same apparatus helps explore geodesics on a lumpy torus:

- Algebraic/symbolic representation

$$\begin{aligned}\text{torus2} = \{\{&(9 + (2 + \mathrm{Cos}[3\,v])\,\mathrm{Cos}[u])\,\mathrm{Cos}[v],\\ &(9 + (2 + \mathrm{Cos}[3\,v])\,\mathrm{Cos}[u])\,\mathrm{Sin}[v],\\ &(2 + \mathrm{Cos}[3\,v])\,\mathrm{Sin}[u]\},\\ &\{u, 0, 2\pi\}, \{v, 0, 2\pi\}\}\end{aligned}$$



- Graphical representation:

(This is an imperfect snapshot from a manipulable Live3D object.)

But, lacking a more general encoding and model of Riemannian geometry, we cannot explore many surfaces of interest in this way.

Example 3, Minimal surfaces

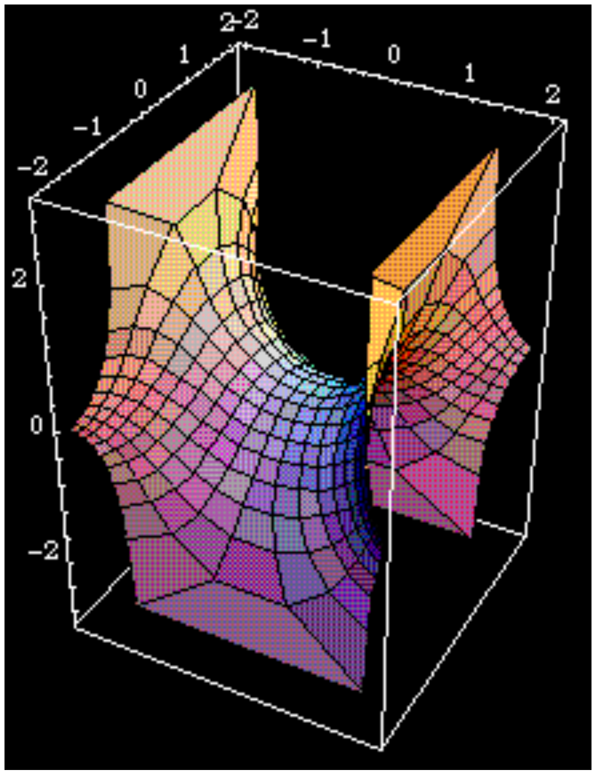
A third example is the Weierstrass representation of a minimal surface via integrals of complex functions

$$\phi(\text{f\_, g\_. z\_}) := \left\{\frac{1}{2}\, f\left(1 - g^2\right),\ \frac{1}{2}\, i\, f\left(g^2 + 1\right),\ f\, g\right\}$$

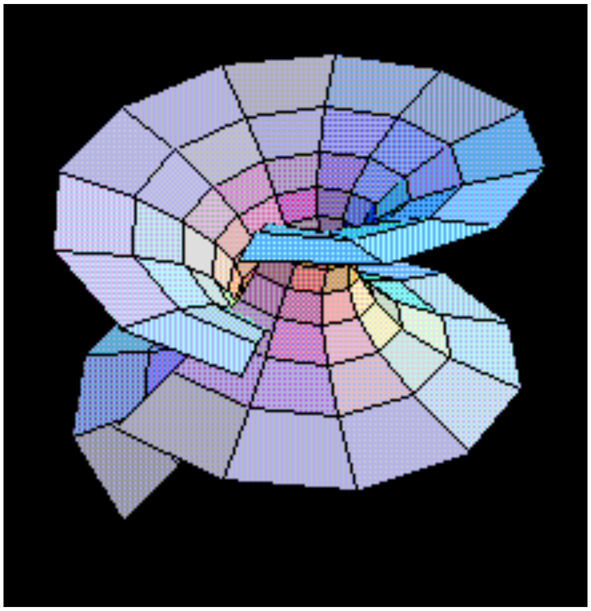
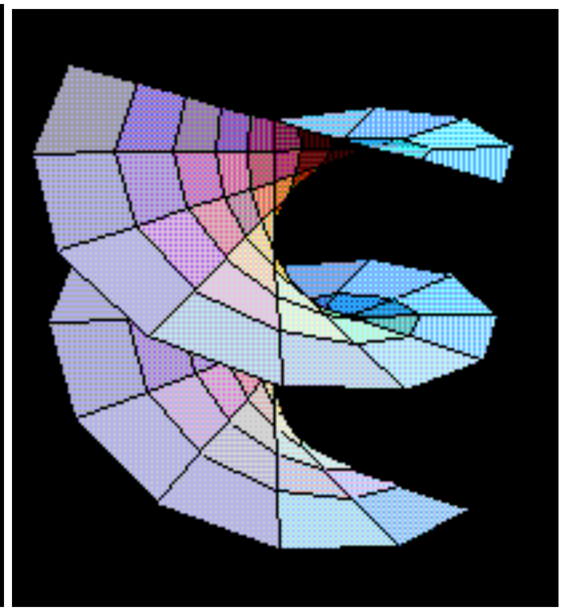
$$W(\text{f\_, g\_, w\_}) := \text{Re}\left(\int_0^w \phi(f,\, g,\, z)\, dz\right)$$

which we can use to explore specific examples:

```
scherk[x_,y_] = W[4/(1-#^4)&, #&, w] /. {w -> x + I y}
```



Here are frames from a movie of a transformation of a minimal surface through a smooth family of isometries. Given the structural representation, it is easy to perform the same "rotation" on any minimal surface.

Example 4, Einstein equation

[Another example is the derivation of the Einstein equation as the Euler-Lagrange equation of the integral of scalar curvature, using a tensor computation system.]

Example 5, Geometry of Lie groups

[Convexity of nilpotent ball (example from R. Karidi):

Given the structure equations for a basis of left-invariant vector fields for a nilpotent Lie algebra of dimension 4 (realized in R^4), study the corresponding geodesic equations.   One goal is to study the shape of the geodesic ball.   This example may be realized in a future draft.]

Examples 5 & 6, Media structures

There are several applications outside mathematics, too. Here are two speculative examples, the first conceptual, the second design.

• More flexible models of "time" -- alternatives to the interval as a parameter for "time-based" media.   Conventional time-based editors use a notion of parallel streams of data, all synchronized to a single parameter.   Other models use a notion of a tree of clocks, which may be difficult to understand for lack of conceptually simple abstractions (like a "loop").   Another problem with the conventional model of time is that it assumes an algebra of intervals which can be complicated and difficult to use, again because of the explicit parametrization of media by intervals. (For an example, open a QuickTime movie on your Macintosh or PC under a common editor or player.)   Here, very simple notions from geometry and analysis may be useful.

• More flexible models of "space" -- one issue now bedeviling virtual reality architects is the problem of coordinatizing symbolically defined “spaces” which are created by independent rules and models. To a geometer, their problem of making “transitions” between local coordinate systems falls neatly into the notion of a manifold. Whereas *ad hoc* solutions worked for planar domains, architecting these more complex and possibly dynamically varying “spaces” will need well-founded geometric models. This is an area ripe for the application of notions from differential geometry and topology.

V. Appendices

Related Work

How is the GW novel? Let me survey existing mathematical computation and writing systems and point out some limitations of each that the GW addresses. The deep sense in which the GW is novel is that the GW is a system for *creating geometric methods and structures*, not a system that purports to autonomously perform geometric reasoning.

1. Blackboard

Although in many respects, blackboards and paper remain the ideal media of exploratory mathematical writing, they lack computational support and memory. In fact, we can use blackboard and paper media as the reference media and use their qualities (cheap, fast, non-volatile, "free" syntax) as a guide for evaluating computational media.

2. Visualization

A general and serious constraint in many visualization systems is that their data structures are restricted to represent, for example, polyhedral complexes in R3, or a small set of graphics primitives like line segment, disc, or piecewise cubic polynomial curves, or a tree of graphics primitives.

In *Geomview* data structures are much too primitive (eg MESH, polygon list) -- with no differential geometric structure. Operators are assumed to be of fixed type: eg. affine transformations of R^i (i = 2,3,4), or coded in a primitive, C-like, programming language in which it is difficult to read the mathematical sense from the code. Applications rely on peculiarities of H^3 and R^3. *Geomview* was designed for a peculiarly powerful set of software and hardware -- Silicon Graphics / GL graphics language -- oriented mainly to rendering a particular class of 3D graphics, rather than manipulating topological or differential geometric structures. Consequently there is a plethora of texture-mapping and lighting and remarkable navigation/positioning functions, but a paucity of any general geometric functions. For example, there is a Hinge tool, written specifically for polyhedra in R^3 or H^3, which allows the user to rotate a face of a polyhedron around a designated edge. Such a function does not serve many geometric research needs, and in any case had to be written directly in a low-level programming language.

A more recent invention, *Pisces'* principal contribution is the drawing of curves and surfaces defined as levelsets of piecewise smooth functions. Otherwise there is no symbolic or graphical manipulation environment. It uses *Geomview* for 3D manipulation, and uses a hodge-podge of control panels written in TCL. These control panels use an instrumentation panel design metaphor which adds an obscuring layer of metaphor atop the mathematician's gestural process.

C. Gunn's *Oorange* is perhaps one of the most powerful of current generation 3D graphics visualization systems. However, its most salient features -- dynamically linkable object-oriented programming units, and an user-interface operator tree -- assume a very elaborate programmer's mental model, a superstructure which obscures altogether geometric and other forms of mathematical analysis.

3. General programming languages

Inappropriate data structures for geometry complicate the description of geometric operations that must act on such structures. (Algorithms that take advantage of non-trivial structure, such as such as boundary operators defined on k-skeletons are exceptional. Besides, they are usually wasted by being specialized to graphics purposes, like texture mapping, rather than geometric purposes.) Given primitive structures, it is hard to define new structures and maps like computing or estimating an integral using the Gauss Bonnet theorem.

As an illustration of this unnaturality of description, contrast this description of an inner product:

```
u.v
```

with this one:

```
float dotuv[21];
for i = 0 to 20 do
   dotuv[i] = u[i] * v[i];
end
```

Aside from the obvious defects of the second representation: (A) more verbose, (B) extraneous structure (for-loop, semi-colons syntax), (C) assumption on dimension, the most serious disadvantage is that there is a strong assumption of type (*i* is an integer, *dotuv* is a 21-dimensional vector of machine floating point numbers). Type-checking is critical for mechanical computation, but maybe not for an effective description of mathematical structure.

Proofs written in such a low-level description tend to be difficult to read, and consequently errors are more difficult to detect. The usual workaround is to limit programs to very short code segments, but then the mathematician must specify computational steps in a verbose and weak language, greatly lengthening the entire exposition. More subtly, the logical relation between an algorithm written in pseudo-code and proofs about the algorithm can be uncertain when the translation, done in an ad hoc fashion, may be faulty. See for example, an early preprint by Hass and Schlafly on the Double Bubble conjecture (1996). In the pseudocode in [HassSchlafly], it is not clear how a function's return value is to be used in the flow of control of an algorithm. This ambiguity in the representation of the algorithm was not reflected in the proof of mathematical correctness, and more importantly, *could not* be represented in the proof because it relies on semantics ("return value of function", "flow of control across multiple subroutine calls") which are outside the semantics of a proof in geometric analysis. In the GW, this would be obviated by casting this description in a higher-order language which could be used to write a proof as well as an algorithm.

The problem of type is one of the most subtle and practically intractable issues in the theory of programming languages. [] This is a concept analogous to mathematical concepts of a function's domain and range, but in a more primitive level. Systems such as *AXIOM* or programming languages like C that make type very explicit tend to be extremely verbose and too cumbersome by the standards of efficiency characteristic of mathematician's English.

Using one of the more sophisticated type-sensitive languages like ML, we can define domains and functions:

```
type Quaternion h
type Quaternion-> Quaternion f
type QStarNorm = (Quaternion-> Quaternion) -> Real T
```

In fact, algorithms have been implemented in ML to perform category-theoretic operations on algebraically defined mathematical structures. There is a trade-off between the precise algebraic power of systems in which type is carefully and explicitly tracked, and the flexibility of pattern-rewrite systems in which the mathematician can freely invent new interpretations of syntax. The GW will hew close to this latter, flexible end of the spectrum.

4. Computer algebra

Most of the leading computer algebra systems are not powerful enough for numeric simulations, and are not integrated with standard numerical analysis packages.

AXIOM is exceptionally powerful and general, but the algebraic machinery makes it cumbersome to use. Nonetheless, the powerful domain definition facilities make it a useful reference computational algebra system. *Ricci* is a very powerful computation system for tensors on bundles, but is a purely algebraic engine, not connected with an explicit metric in a way which would make it usable in conjunction with analysis or numerical estimates.

A problem common to many of the more popular symbolic manipulation programs, including *Maple's* and *Mathematica*'s core algebra engines, is that they are designed to *automate* computation. A practical consequence of this design is there are opaque functions that are inaccessible to the working mathematician because they are compiled or written in low-level code (machine language, or C-like code). This is an unavoidable necessity if we are to build systems that work reasonably fast. The argument that all code must be equally transparently inspectable by the user ignores the socio-textual fact that readers of analog scientific literature always have a textual horizon -- they can follow citations only to a certain depth. But we can usefully distinguish inspectability-in-principle from uniform inspectability. Some degree of inspectability-in-principle is certainly necessary to sustain scholarly discipline, and this will be true for the geometric models that will be described by the GW. This will necessarily be the case, as we propose to write a novel descriptive and interpretive layer for geometric structures that will make it much easier to use sketching and direct manipulation interfaces.

5. Sketching systems

Computer graphics and interface researchers are beginning to develop direct-manipulation environments in which a user can create graphical elements with freehand gestures rather than going through an interpreter of written coded descriptions. (Imagine being forced to write an integral sign -- the sigma -- by specifying it as a parametrized curve rather than simply making a stroke.) Perhaps the most ambitious commercial operating system built around gestures was the Penpoint operating system developed by GO Technologies in the 1980's []. GO and later gesture-recognition systems have had limited success because of the complexity of the full range of human gestures used in writing on a flat plane. Some recent and current projects have had notable success by simplifying the problem in several ways, by making "shallow" systems that merely capture bits without immediate (deferred) interpretation (eg. *Tivoli* -- T. Moran), or by restricting to special contexts. These include *Geometer's*

Sketchpad, Napkin and *Sketch*. *Geometer's Sketchpad* is restricted to ruler-and-compass 2D geometry. In some sense, one might view the GW as an attempt to generalize *Geometer's Sketchpad* to differential geometry. *Napkin* is by design similar in spirit to the GW, in that it aims to augment the earliest stages of free form exploration. It is designed for architects' freehand sketching. Similarly, Zeleznik & Hughes et al's *Sketch* is designed to support mechanical engineers' initial sketching with significant ambiguity in the interpretation of their strokes. But the interpretive model used by *Sketch* is nonetheless a relaxation of the CAD structures and notions, which makes it unsuitable as it stands for mathematical creation. Similarly with the other systems. Typically these systems use a fixed palette of objects and operators, with interfaces that fall into the fixed structure paradigm which makes it extremely difficult to adapt them for uses by geometers and others outside the disciplines for which they were originally conceived.[23]

By basing feature-recognition on a sophisticated, but limited geometric model, the GW attempts to reduce difficult problem of gesture recognition to a tractable one. Results may be incorporated in a new generation of reactive environments. Recently, T. Winograd, P. Hanrahan and students have begun construction of a wall-sized high-resolution displays with a software architecture that may accomodate novel input methods.[24]

6. General Hybrid Math Systems

Mathcad, the most ambitious of hybrid mathematical computation systems, despite some novel interface hybridizations, suffers from a confused integration of programming language, numerical algorithms, and interface structures. Many features which properly should be part of the symbolic algebra or of a programming language (eg. Transpose, Simplify<Evaluate) are wired into the interface as menus. This means that the program treats certain mathematical operators as atomic functions, which cannot be changed or integrated into user-defined operations. Structured layout is based on the model of a coarse tree of graphics rectangles, which is too cumbersome.

7. Numerical Simulation

Matlab, arguably the most popular numerical matrix analysis program, is limited fundamentally to two-dimensional arrays of real or complex floating point numbers. Therefore all structures must be cast somehow into such representations in order to apply Matlab functions. K. Brakke's *Surface*

23 Consider how much broader are the communities of mathematicians who use paper and chalkboard.

24 T. Winograd, "A Human-Centered Interaction Architecture," 1998.

Evolver [Brakke] lies at the other extreme -- it was hardwired specifically to evolve piecewise surfaces according to the minimization of certain functionals dependent on the normal and the curvature. By comparison, the GW will make it possible to quickly and reliably construct special numerical models within an integrated system.

8. Socio-technical Practice

With the developing sophistication on the part of designers of complex or powerful computer systems since 70's and 80's came the realization that socio-economic dynamics had to be considered in the design of computation that could be used for "real work" by non-programmers. In this context, the GW represents an attempt to go beyond the "Hollywood model" of scientific computation that we've inherited from the era of supercomputing centers. In this model of computationally augmented research, a senior researcher acquires a team of technical experts and artists who build a special system using special hardware or software to explore some structures of particular interest to this researcher and his/her school. While this was appropriate when computation was expensive and rare, we now have the opportunity to take advantage of much more ubiquitous powerful computation, and even more important, of a significantly richer software framework for intercommunicating computational processes, and more expressive languages than those commonly used by non-mathematical programmers. This decentralized, finer-grained use of technology may fit better the practices of the contemporary mathematical research community.

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