
Society for Literature and Science

Geometric Measure Theory As a Monster Mash

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Simpler Question

What is the Area of a Surface?

Before going on with the Plateau problem, let's warm up with "simpler" prototypical question:

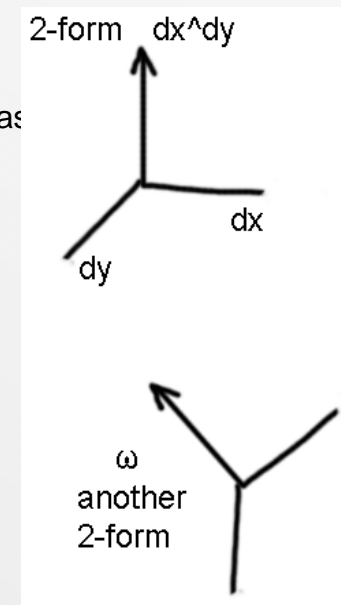
What is the Area of a Surface?

1. What is area?
2. What is surface?

For **smooth** (continuously differentiable) surfaces, we have a classical way of measuring area:

via area 2-forms, such as $dx \wedge dy$, as area meters

(like light meters, sensitive to both position and orientation).

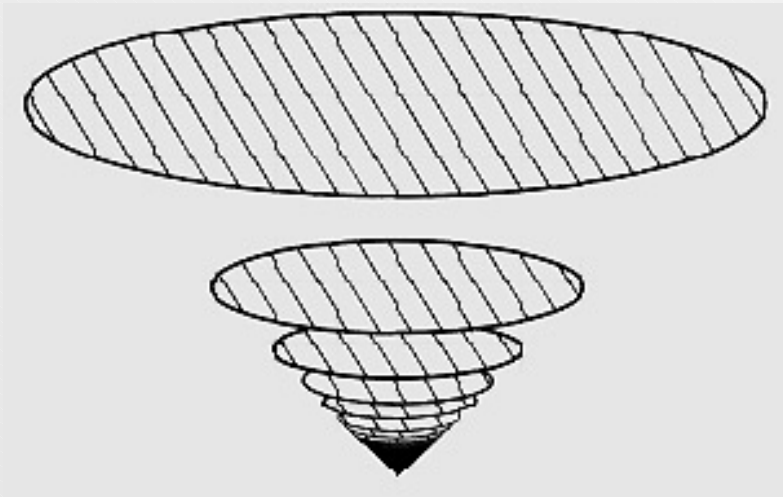


Monstrous Surfaces

Lipschitz graph: $|f(x) - f(y)| \leq C|x - y|$

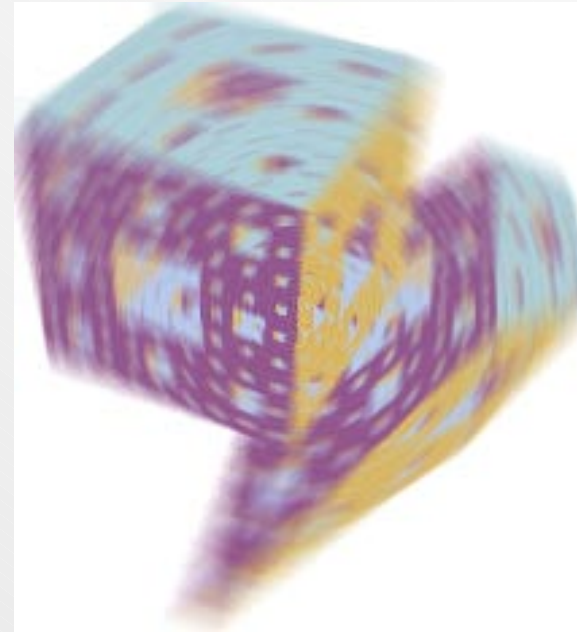


Countable unions of Lipschitz graphs.



finite area, infinite boundary (!)

Iterated sets.

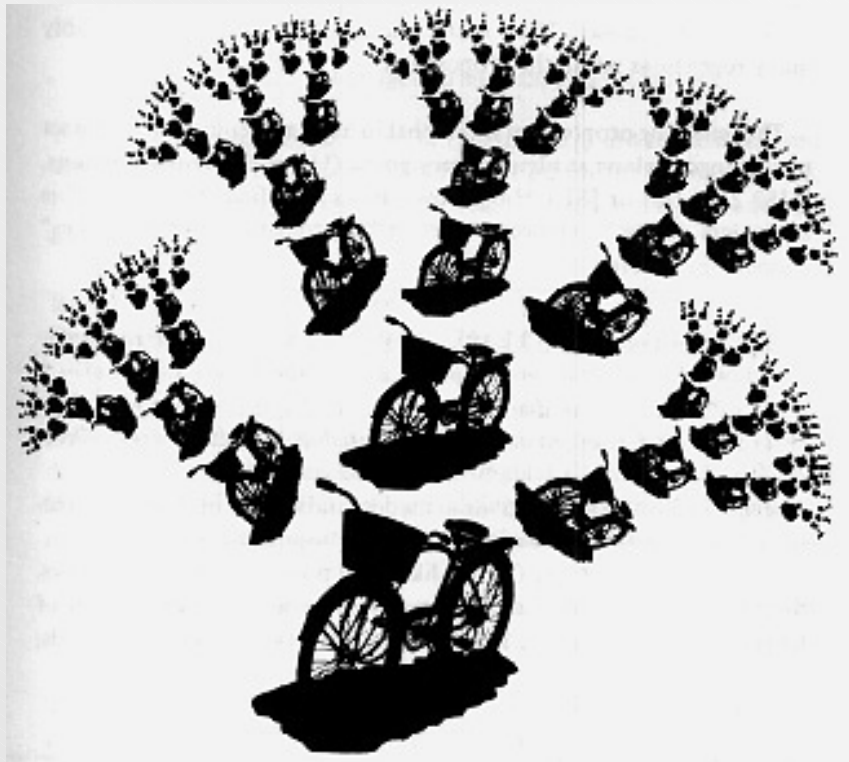


Sierpinski sponge.

Area Fails

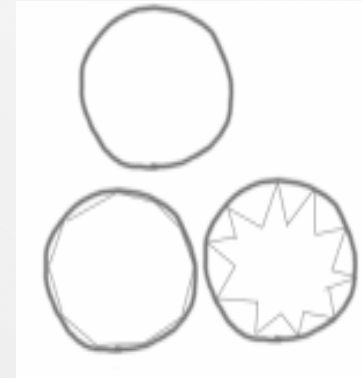
dx^dy ?? Approximation by Lipschitz graphs??

Countably rectifiable

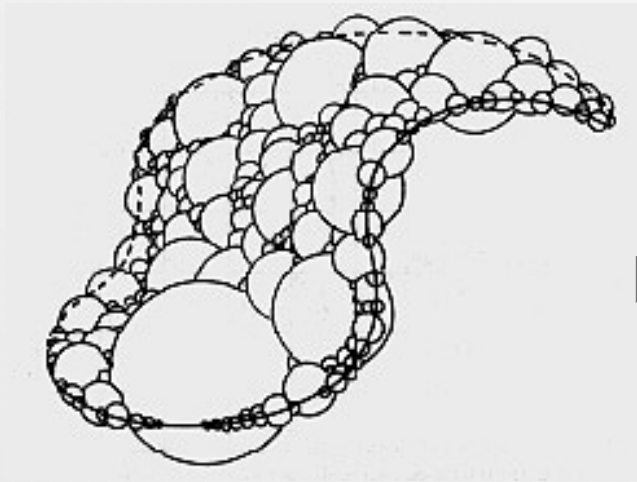


space-filling limit (!)

Even a cylinder



Extend the Notion of *Area*



Hausdorff Measure H^m

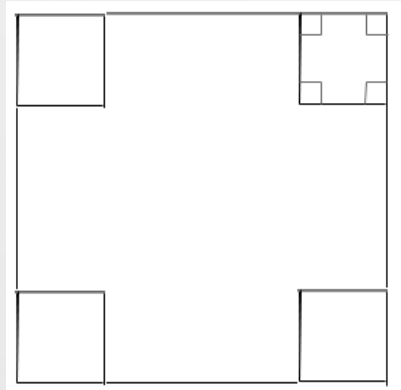
$$H^m(M) = \lim_{\delta \rightarrow 0} \inf_{M \subset \bigcup B(r), r < \delta} \left(\sum_B \frac{\mu(B^n(r, x) \cap M)}{\alpha_m r^m} \right)$$

Another Extension of *Area*

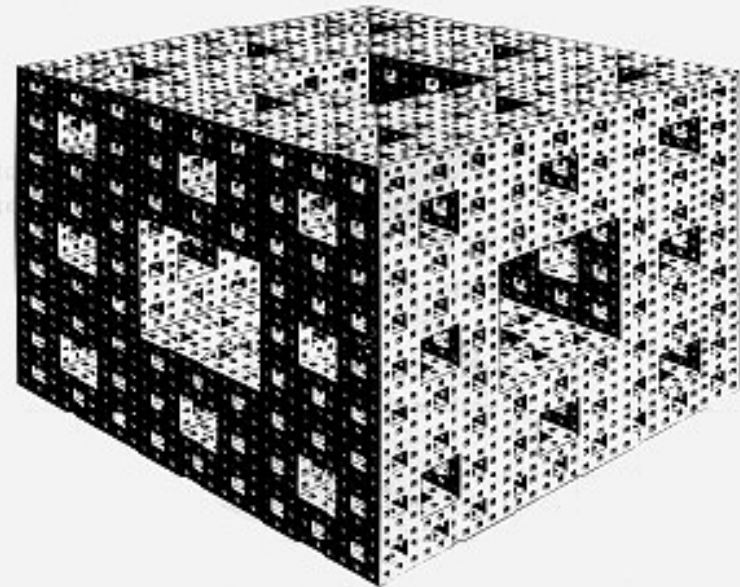
Projection (Favard) Measure

Project into hyperplanes

Monster: There are sets that are thin in almost every direction - but have positive Hausdorff measure.



This set has H^1 measure $\sqrt{2}$
but I^1 measure 0



(Again) Plateau's Question

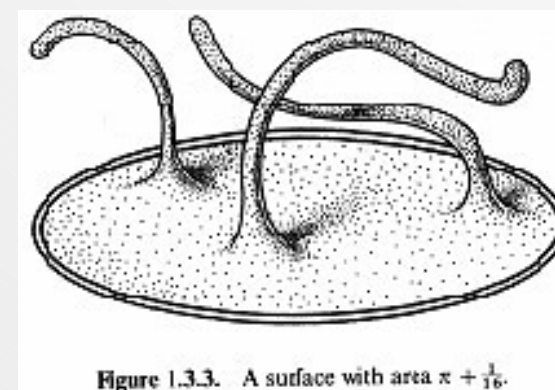
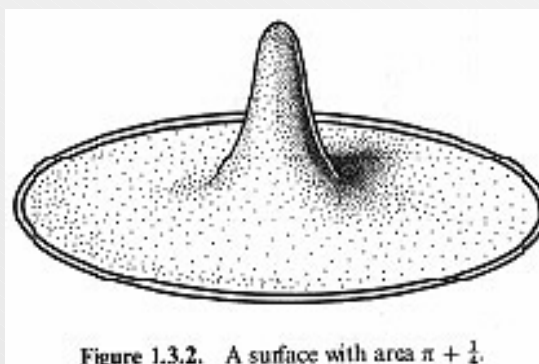
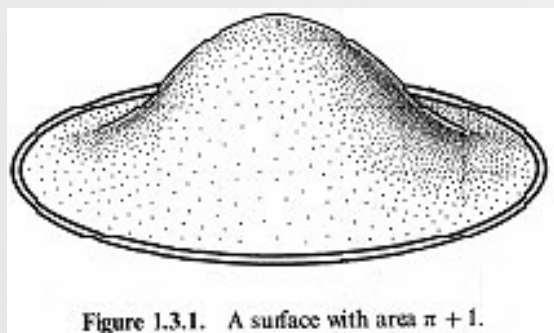
The Minimizing Sequence: $|M_1| > |M_2| > |M_3| > \dots ?$

Armed with these hybridized notions, let's go back to the Plateau question

Return to characterization of minimality: The Minimizing Sequence:

Take ALL surfaces that span the given loop, and pick out the one that has least area.

But, here's a pathological sequence:



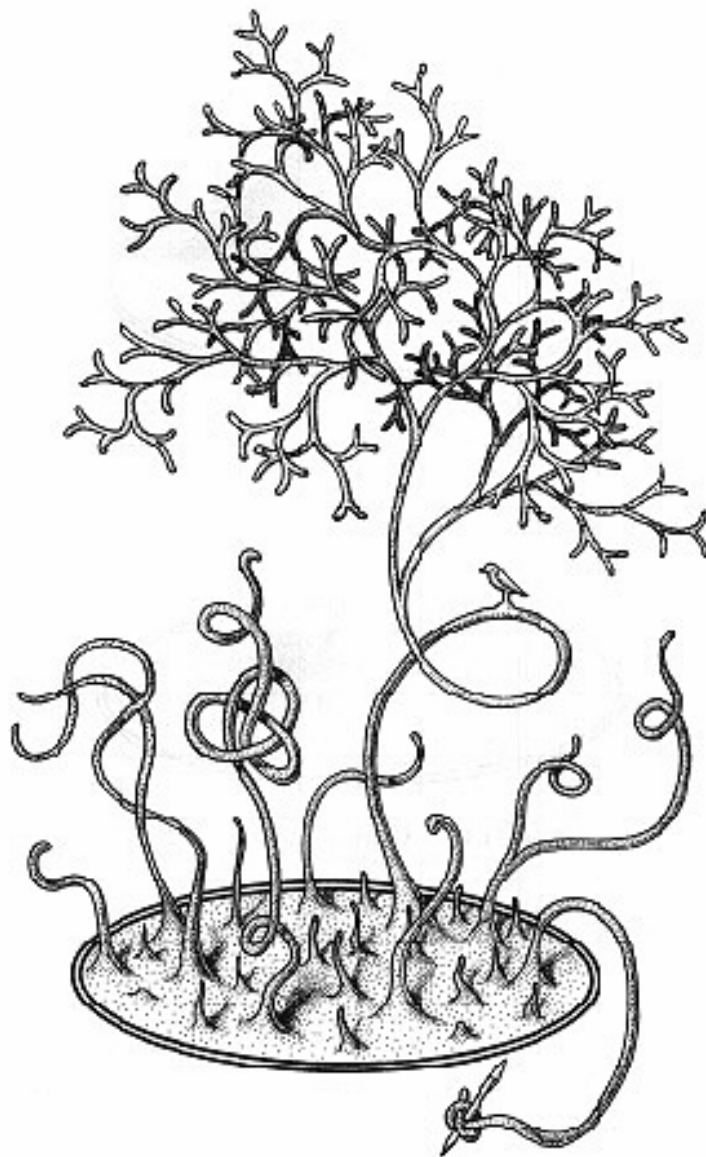


Figure 1.3.4. A surface with area $\pi + \frac{1}{4}$.

(Again) What Is a Surface?

Extend Notion of Surface to *Currents*:

$$\cancel{A[M, d\omega]} := \int_{\cancel{M}} \cancel{d\omega}$$

$$A[_] := \int_{\cancel{_}} \cancel{d\omega}$$

Measures (ex. 2-form on surface)

Currents

$$T[_] := \int_{\text{all space}} d__$$

Measures on measures

Lakatos' Heuristic

Proofs and Refutations

In his virtuoso essay, "Proofs and Refutations," Imre Lakatos made central use of the notion of a counterexample to a conjecture. A conjecture-demolishing counterexample, which Lakatos called a **monster**, was an event that according to his mathematical dialectic, generated new theory in the form of modified statements and concepts.

He described his heuristic fairly clearly, for example in [Appendix 1]:

1. Make a primitive conjecture.
2. Find global counterexample C.
3. **Proof Analysis**: Isolate part of proof that's contradicted, and make it a lemma with C as its local counterexample.
4. Add newly explicit lemma to the statement of the new conjecture.
- ...
5. If new proof-generated concept or lemma appears in many theorems, then it becomes promoted as more central to the theory.
6. Counterexamples provide new avenues of inquiry and are most important in new, young, rapidly developing fields.

Lakatos habilitated monsters - or counterexamples - into the normal course of mathematical work, but at a rather formal, rhetorical, level, that of the level of proofs. It's telling that, despite his nod toward discourse, he focussed the central part of his discussion of his method on what he termed proof analysis.