The Willmore Functional and Möbius Strips

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5 January 1993

1 Introduction.

Let ϕ be an immersion of a surface M into R^3, $\mathbf{H}$ the mean curvature vector on ϕ[M], $*1$ the area 2-form induced by ϕ, and define the total mean curvature functional W by

$$W[\phi] = \int_M |\mathbf{H}|^2 * 1 \tag{1}$$

Given a Möbius strip M isometric to a fixed rectangle R, characterize the flat (Gauss curvature $K = 0$) immersion $\phi : M \to \mathbf{R^3}$ which minimizes W.

The physical analog for this problem is to characterize the half-twisted coil of paper (Möbius strip) which has least bending energy. This problem differs from the classical Willmore problem in three respects: (i) M has a boundary, (ii) M is non-orientable, and (iii) M should be flat. There are actually two parts to this problem. One is the existence and regularity of a minimizer which is the main subject of this note.

Theorem 1.1

The infimum of W is achieved by a smooth Möbius strip.

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The more interesting part is to find a geometric characterization of the minimizer.

(i) requires some extension of a few estimates in [Simon]. Certain methods from L. Simon's existence proof [Simon] may be applied to this problem. The existence theory seems to be insensitive to orientation so (ii) should be important only to rule out a cylinder. The flatness condition (iii) makes it difficult to apply variational arguments which assume unconstrained 1-parameter families of surfaces.[1]

In section 1 we discuss the geometry of Möbius strips, in section 2 we discuss the measure-theoretic limit, in section 3 we study regularity of the limit. In section 4 we take a different tack and set up a classical variational problem on closed curves in R^3. In the last section we present some experimental findings using a generalized Hamilton-Jacobi level-set evolution scheme.

2 Geometry of flat Möbius strips.

For a fixed $L > 0, b > 0$ define coordinates (u,v) on the rectangle $R =$ [0,L] x [-b, b]. Let $\phi : R \to \mathbf{R^3}$ be an isometric immersion of the rectangle onto a flat surface M. Note that the curve $\sigma(u) \equiv \phi(u, 0)$ is a geodesic in M. We call σ the *spine* of the Möbius strip. Flat surfaces are unions of ruled surfaces [2] , so we may alternatively parametrize $\phi[R]$ by the map

$$\tilde{\phi}(s,t) = \sigma(s) + t\xi(s) \tag{2}$$

where $s \in [0, L]$ is the arclength parameter of σ, $\xi(s)$ is the unit ruling vector, and $t \in [\frac{-b}{\sin(s)}, \frac{b}{\sin(s)}]$ is the coordinate along a ruling which passes through the point $\sigma(s)$. $\xi(s)$ can be thought of as the 0-eigenvector of the second fundamental form at $\tilde{\phi}(s, t)$. We will call (s,t) coordinates *foliation coordinates.*

Let κ and τ denote the curvature and torsion of σ, and let $\{T, N, B\}$ be the Frenet frame along σ. Note that, since σ is a geodesic, N is parallel to

[1] However, by a general theorem on integral systems of pde's, if the boundary is allowed to vary freely, any flat surface can be embedded in a 1-parameter family of flat surfaces. [L. Hsu, personal communication]

[2] Discussions of the geometry of flat ruled surfaces can be found in [Spivak5], [Klingenberg]. *expand later*

ν, the unit normal to M. Indeed we can choose $N \equiv \nu$ along the spine. For a Möbius strip, the boundary conditions are

$$\left\{ \begin{array}{c} \sigma(0) = \sigma(L) \\ \xi(0) = -\xi(L). \end{array} \right. \tag{3}$$

Note that along a ruling, the tangent plane is constant.

Also, there exists an $s_0 \in [0, L]$ such that $N(s_0) = -N(s_0 + L)$, the curvature κ vanishes, otherwise we could extend N along the rulings to a global continuous unit normal field on M. We may without loss of generality choose $s_0 = 0$. Let $\theta(s)$ be the angle between $\xi(s)$ and the tangent vector to $\dot{\sigma}(s)$. Each ruling must be the image of a straight line segment in R, since ϕ is an isometry. It follows that there is constant $\alpha(b, L) > 0$ such that

$$\forall s, |\frac{\pi}{2} - \theta(s)| \geq \alpha. \tag{4}$$

This simple inequality is useful because it does not depend on the immersion.

(u,v) and (s,t) are related by the following coordinate transformation:

$$\left\{ \begin{array}{c} v = t\sin(\theta), \\ u = s + t\cos(\theta) \end{array} \right. \tag{5}$$

and the area form

$$*1 = dudv = \sin(\theta)(1 - \beta t)dsdt \tag{6}$$

where $\beta = \frac{\dot{\theta}}{\sin(\theta)}$. (Throughout this note, $\dot{} \equiv \frac{\partial}{\partial s}$.) Note that along the spine $t = 0$ and $u \equiv s$. From nondegeneracy of the metric it follows that $\dot{\theta} < \frac{\sin(\theta)}{t}$.

The Gauss and the mean curvatures are given respectively by

$$K = det(A);\ H = \frac{1}{2}trace(A) \tag{7}$$

where A is the second fundamental form. In matrix form,

$$A = G^{-1}B \tag{8}$$

where G = the metric g_{ij}, $B_{ij} = \frac{1}{\sqrt{G}} < \nu, \phi_{,ij} >$, and $\phi_{,ij}$ = the second partials of ϕ.

Along a ruling, *i.e.* for a fixed s, the mean curvature function $H = |\mathbf{H}|$ satisfies

$$H = \frac{H_0}{1 - \beta t}$$

where H_0 is the mean curvature at $\sigma(s)$. By the Euler formula,

$$H_0 = \frac{\kappa}{\sin^2(\theta)}.$$

Lemma 2.1

Let $\tilde{\phi}$ be a Möbius strip parametrized as in (2). Then $\tilde{\phi}$ is flat iff

$$< T, \xi, \dot{\xi} > \equiv 0, \tag{9}$$

i.e. $T, \xi, \dot{\xi}$ *are coplanar for* $\forall s$.

Proof. The proof is a calculation from the parametrization of the mobius strip ((2)) and the definitions ((7)) and ((8)).

Representing the normal ν as $\nu = \frac{\dot{\phi} \wedge \xi}{l}$, where $l \equiv \|\dot{\phi} \wedge \xi\|$, we have the second fundamental form

$$A = \frac{1}{l} G^{-1} \begin{bmatrix} < \nu, \phi_{ss} > & < \nu, \phi_{st} > \\ < \nu, \phi_{ts} > & < \nu, \phi_{tt} > \end{bmatrix} = \frac{1}{l} G^{-1} \begin{bmatrix} < (\dot{\sigma} + t\dot{\xi}) \wedge \xi, \ddot{\sigma} + t\ddot{\xi} > & < (\dot{\sigma} + t\dot{\xi} \wedge \xi, \dot{\xi} > \\ < (\dot{\sigma} + t\dot{\xi} \wedge \xi, \dot{\xi} > & 0 \end{bmatrix} \tag{10}$$

so the Gauss curvature is given by

$$K = -\frac{1}{gl^2} < (\dot{\sigma} + t\dot{\xi}) \wedge \xi, \dot{\xi} >^2 \tag{11}$$

$$= -\frac{1}{gl^2} < \dot{\sigma} \wedge \xi, \dot{\xi} >^2 . \tag{12}$$

where $g \equiv \det G$ Thus the developable surface is flat iff

$$< \dot{\sigma}, \xi \wedge \dot{\xi} > \equiv < T, \xi, \dot{\xi} > \equiv 0.$$

Lemma 2.2 *Let $\tilde{\phi}$ be a flat Möbius strip. Then*

(i) $< B, \xi >= \sin(\theta)$, *and*

(ii) $\frac{\kappa}{\tau} = tan(\theta)$, *and*

(iii) $\|\dot{\xi}\| = \|\dot{\theta}\|$, *and*

(iv) $\|\dot{\nu}\|^2 = (\kappa^2 + \tau^2)$.

Proof. (i) follows from Lemma 2.1, the fact that $\xi \perp N$.

Differentiating

$$\xi = \cos(\theta)T + \sin(\theta)B$$

we obtain

$$\dot{\xi} = -\dot{\theta}\sin(\theta)T + \dot{\theta}\cos(\theta)B + (\kappa\cos(\theta) - \tau\sin(\theta))N.$$

By Lemma 2.1, $< \dot{\xi}, N >= (\kappa\cos(\theta) - \tau\sin(\theta)) = 0$, which implies (ii), and, by taking norms, we get (iii).

Since flatness implies $N \equiv \nu$, as functions of s, (iv) follows from the Frenet equations.

The Willmore functional $W[\phi]$ can be rewritten

$$W[\phi] = \int_M |H|^2 * 1$$

$$= \int_M |\frac{H_0}{1-\beta t}|^2 * 1$$

$$= \int_M |\frac{\kappa}{\sin^2(\theta)(1-\beta t)}|^2 * 1.$$

By a change of variable to (s,v) coordinates, $t = \frac{v}{\sin(\theta)}$ using ((5)), we obtain

$$\int \frac{\kappa^2}{\sin^3(\theta)}(\frac{1}{1-\beta t})\, dtds = \int_0^L \int_{-b}^{b} \frac{\kappa^2}{\sin^4(\theta)}(\frac{1}{1-\gamma v})\, dvds$$

where $\gamma = \frac{\dot{\theta}}{\sin^2(\theta)}$.

$$(13) \quad W[\phi] = \int_0^L \frac{\kappa^2}{\gamma \sin^4(\theta)} \ln(\frac{1+b\gamma}{1-b\gamma}) ds = \int_0^L \frac{\kappa^2}{\dot{\theta}\sin^2(\theta)} \ln(\frac{1+b\gamma}{1-b\gamma}) ds$$

The non-negativity of the integrand requires that $\dot{\theta}$ satisfy:

$$0 \leq \frac{b\dot{\theta}}{\sin^2(\theta)} < 1.$$

Alternatively, using the fact that $\frac{\kappa}{\tau} = tan(\theta)$, (Lemma 2.2), we can write $W[\phi]$ as

$$(14) \qquad W[\phi] = \int_0^L \frac{(\kappa^2+\tau^2)^2}{\kappa^2\gamma} \ln(\frac{1+b\gamma}{1-b\gamma}) ds,$$

which expresses the total curvature in terms of the geometry of the spine, the angle the ruling makes with respect to that curve, and its derivative $\dot{\theta}$. This will be useful in comparing the total oscillation of the normal with the total curvature over a subset of a Möbius strip lying between two rulings.

3 Existence.

We take a minimizing sequence of flat Möbius strips $M_k = \phi_k[R]$, each isometric to the fixed rectangle R. Assume that the images all contain a fixed point $p \in R^3$. We may consider these surfaces in the space of integer multiplicity rectifiable varifolds, *i.e.* Radon measures associated with functionals on the space of two-forms $\Omega^2\mathbf{R^3}$, with compact support. [3] The general compactness theorem for rectifiable varifolds implies a measure-theoretic limit since the strips all have the same area, and their boundaries have fixed length. First we describe our setting.

Let U be an open set in R^n (for our purposes, $n = 3$). Let R_m denote the space of rectifiable m-varifolds, that is H^m-measurable subsets with associated H^m-measurable density functions Θ.

[3]We cannot appeal to the classical theorem that a group of isometries of a connected, locally compact metric space M is locally compact [Theorem 4.7, Kobayashi I, p. 46], because we do not have self maps of a fixed metric space.

Define the norm of an m-varifold T by

$$\|T\| = \sup\{\mu_T[V] + \|\delta T\|[V] : V \subset\subset R^n\}.$$

A varifold T has *locally bounded first variation* if $\|\delta T\| < \infty$. The total variation measure of T, $\|\delta T\|$, is defined in [Simon2, p. 234]. *explicate later.* These are the natural generalizations of unoriented surfaces with finite area and perimeter for which one obtains a compactness theorem.

We state a compactness theorem for rectifiable varifolds without proof:

Theorem 3.1 (Simon2, p. 247.)

Let $\{T_i\}$ be a sequence of integer multiplicity rectifiable varifolds such that $\|T\|(V) < \infty \, \forall V \subset\subset U$. Assume that the associated densities ≥ 1.

Then there is a subsequence $T_{i'}$ and a rectifiable varifold T with locally bounded first variation such that $T_{i'} \to T$ as Radon measures, T has density $\geq 1 \, \mu_V - a.e.$, and

$$\|\delta T\|(V) \leq \liminf_{i \to \infty} \|\delta T_i\|(V), \forall V \subset\subset U.$$

Proof. We refer the reader to [Simon2] for a description of the proof.[4]

From this we can obtain the following

Proposition 3.2

Let M_k be a W-minimizing sequence of flat Möbius strips. Then

$$\liminf_{k \to \infty}(M_k) = M_\infty \in \Omega_2(\mathbf{R^3})_*$$

as varifolds.[5] *Moreover, $M_k \to M_\infty$ in Hausdorff distance.*

[4] It would be interesting but not necessary, I believe, to prove a boundary version of the diameter–area–F estimates contained in Simon's lemma 1 to establish the existence of a measure-theoretic W-minimizing limit for surfaces with boundary. We will study exactly where the hypothesis $\partial M = \phi$ is used in Lemmas 1 and 3, and try to extend them to surfaces with boundary. We will examine the boundary term which appears in the first variation identity: For $\Omega \subset M^m (\subset R^{m+1})$, and $X \in \Gamma T\mathbf{R^{m+1}}$,

$$\int_\Omega div_M X = -\int_\Omega < X, \mathbf{H} > - \int_{\partial\Omega} < X, \eta >$$

where η is the unit inner conormal to the boundary $\partial\Omega$.

[5] We would like to prove that M_∞ has unit density, as well.

Proof. *deferred.*

The existence of the varifold limit follows directly from the varifold compactness since the M_k all have identical finite area and boundary measures. We argue that (1) the limit has a generalized tangent plane a.e. and therefore M_∞ is rectifiable, (2) $M_k \to M_\infty$ in Hausdorff distance, and (3) the limit M_∞ is ruled, where each ruling is a limit of lines in M_k. (See the discussion in section 2.)

We first show that the minimizing surfaces approach a limit set as point sequences in R^3.

Lemma 3.3

Let $p_k \in M_k$ where M_k is a W-minimizing sequence of connected surfaces converging to a surface M_∞ as varifolds. Then $p_k \to p \in M_\infty$.

Proof. Assume that the $p_k \to p$ where $p \notin M_\infty$. We will use the monotonicity theorem to show that $lim_{k\to\infty} W(M_k) = \infty$. The argument is essentially identical to [Simon, Toro]. We can use the monotonicity theorem to prove a local lower bound of the form:

$$(*) density + \int H^2 \geq \pi.$$

The idea is to apply (*) to finite disjoint unions of patches on the M_k, near the p_k let $k \to \infty$, and contradict

$$W(M_\infty) < \infty.$$

The divergence theorem states that, for a vector field X on a surface M, and an open set $\Omega \subset M$,

$$\int_\Omega div_M(X) = \int_\Omega < X, H > \tag{15}$$

Let ρ and σ be fixed constants $\rho > \sigma > 0$, let $y \in \Omega$ and let $r(x) = |x - y|$ be the distance function in R^3.

Choosing

$$X(x) = \left(\frac{1}{\rho^2} - \frac{1}{r_\sigma^2}\right)_+ (x - y)$$

where $u_+ \equiv max(u,0)$, we derive the following consequence *[Lemma (())]* of the monotonicity identity:

$$\left(1+\frac{\alpha}{4}\right)\left(\frac{|\Sigma \cap B_\rho(\xi)|}{\rho^2}\right)+\left(\frac{1}{4\alpha}+\frac{1}{16}\right)\int_{\Sigma\cap B_\rho(\xi)} \|\mathbf{H}_{\mathbf{\Sigma}}\|^2 dH^2 \tag{16}$$
$$\geq$$
$$\left(1-\frac{\beta}{4}\right)\left(\frac{|\Sigma \cap B_\sigma(\xi)|}{\sigma^2}\right)+\left(\frac{-1}{4\beta}+\frac{1}{16}\right)\int_{\Sigma\cap B_\sigma(\xi)} \|\mathbf{H}_{\mathbf{\Sigma}}\|^2 dH^2$$

where $\sigma < \rho$ and $\alpha, \beta > 0$.

If M_k does not converge to M in Hausdorff distance then there exists a ρ such that $B_{2\rho}(p) \cap \partial M = \phi$. Assume that k is sufficiently large so that $|p_k - p| < \rho$.

Figure 1.

Fix an integer N and choose r so that $Nr = \rho$. Since M_k is connected, there is a point $\eta_i \in M_k$ in each shell $B_{ir}(p_k) \setminus B_{(i-1)r}(p_k)$ with $B_{r/4}(\eta_i) \subset B_{ir}(p_k) \setminus B_{(i-1)r}(p_k)$.

Figure 2.

We estimate W over the topological disks $M_k \cap B_{r/4}(\eta_i)$. We apply the monotonicity inequality ((16)) with $B_{r/4}(\eta_i)$ on the left hand side, and let $\sigma \to 0$ to obtain a lower bound on $W[B_{r/4}(\eta_i) \cap M_k]$:

$$(17)\left(1+\frac{\alpha}{4}\right)\frac{|B_{r/4}(\eta_i)\cap M_k|}{(\frac{r}{4})^2}+\left(\frac{1}{4\alpha}+\frac{1}{16}\right)\int_{B_{r/4}(\eta_i)\cap M_k}\|\mathbf{H_M}\ \|^2 dH^2 \geq \pi.$$

Consider the disjoint union $A \equiv \bigcup_{i=1}^{N} B_{r/4}(\eta_i)$, $A \subset B_\rho(\eta_k) \subset B_{2\rho}(\eta)$. ((17)) implies that

$$(18)\qquad N\pi \leq \left(1+\frac{\alpha}{4}\right)\frac{|A\cap M_k|}{(\frac{r}{4})^2}+\left(\frac{1}{4\alpha}+\frac{1}{16}\right)\int_{A\cap M_k}\|\mathbf{H_M}\ \|^2 dH^2$$

and

$$(19) \quad \leq \left(1+\frac{\alpha}{4}\right)\frac{|B_{2\rho}(\eta)\cap M_k|}{(\frac{r}{4})^2}+\left(\frac{1}{4\alpha}+\frac{1}{16}\right)\int_{B_{2\rho}(\eta)\cap M_k}\|\mathbf{H_M}\|^2 dH^2$$

By our hypothesis

$$\lim_{k\to\infty}|B_{2\rho}(p)\cap M_k| = 0$$

so, choosing $\alpha = 4$, we obtain

$$\liminf_{k\to\infty}\int_{B_{2\rho}(p)\cap M_k}\|\mathbf{H_M}\|^2 dH^2 \geq 8N\pi$$

Since N was arbitrary, we have

$$\liminf_{k\to\infty}\int_{B_{2\rho}(\eta)\cap M_k}\|\mathbf{H_M}\|^2 dH^2 = \infty$$

which contradicts the fact that M_k is a minimizing sequence.

Definition 3.4 *For a surface Σ and a ball $B_\rho(x)$, define $\Sigma_\rho(x) \equiv \Sigma \cap B_\rho(x)$. For simplicity, we will often drop the center x.*

Lemma 3.5

M_∞ *is a union of lines* (up to a set of measure zero).

Proof. (Sketch) For almost every $p \in M_\infty$, there is a sequence of $p_k \in M_k$ which converges to p in R^3. Otherwise, there would exist a ball $B_\rho(p)$ where $(M_\infty)_\rho \cap M_k = \emptyset$ for all sufficiently large k. Let χ be the (C^0 approximation to the) characteristic function on $B_\rho(p)$. On one hand,

$$\lim\int_{M_k}\chi = \lim\int_{B_\rho(p)\cap M_k}\chi = 0$$

but

$$\int_M \chi > 0$$

which contradicts the fact that $M_k \to M_\infty$ as varifolds. Consider the associated line segments $L_k \subset M_k$, $L_k \ni p$. Observe that $|L_k| \geq 2b$ and that for sufficiently large k, there is some $R \geq b > 0$ such that $L_k \subset B_R(p)$. By compactness, a subsequence $L_{k'}$ converges to L_∞, a line segment of length $\geq 2b$ containing p. Moreover, each point on L_∞ is in M_∞ since M_∞ contains the accumulation points of sequences $\{q_k\} \in L_k \subset M_k$.

Let R be the fixed domain rectangle. For an interval $I \subset [0, L]$ and an immersion ϕ define $R_{\{I,\phi\}} \subset R$ by

$$R_{\{I,\phi\}} \equiv \{p \in R : \text{ruling through p meets } I \times \{0\}\};$$

$$R_{\{I,\phi\}} \equiv \{(s,t) : s \in I\},$$

in foliation coordinates. We'll abbreviate $R_{\{I,\phi\}}$ as R_I when the immersion is understood from context.

From equation (14), we define a restriction of the total mean curvature

$$W[I] \equiv \int_{R_I} H^2 * 1 = \int_I \frac{(\kappa^2 + \tau^2)^2}{\kappa^2} \frac{1}{\gamma} \ln(\frac{1 + b\gamma}{1 - b\gamma}) ds \tag{20}$$

we note that it should be possible to estimate the total oscillation of the normal.

In view of Lemma 2.2, define u by

$$u \equiv \|\dot{\nu}\|^2 = \kappa^2 + \tau^2. \tag{21}$$

Lemma 3.6

$\int_I \|\dot{\nu}\|^2 ds \leq CW[I]$, *and*

$\int_I \kappa^2 ds \leq CW[I]$, *where C is independent of I as well as the immersion ϕ.*

Proof.

Note that

$$\int_I \|\dot{\nu}\|^2 ds \equiv \int_I u ds \leq (\int_I (\frac{u}{\kappa})^2 ds)^{\frac{1}{2}} (\int_I \kappa^2 ds)^{\frac{1}{2}} \tag{22}$$

by the Schwarz inequality.

Define

$$P(\gamma, b) \equiv \frac{1}{\gamma} \ln(\frac{1+b\gamma}{1-b\gamma}). \tag{23}$$

Notice that $P(\gamma, b) \geq 2b$. Then

$$\begin{aligned} W[I] &= \int_I \tfrac{u^2}{\kappa^2} P(\gamma, b) ds \\ &\geq 2b \int_I \tfrac{u^2}{\kappa^2} ds \\ &\equiv 2b \int_I \kappa^2(1+\mu^2) ds \\ &\geq 2b \int_I \kappa^2 ds. \end{aligned} \tag{24}$$

where $\mu \equiv \frac{\tau}{\kappa}$.[6] Then we can bound both factors in the right hand side of ((24)) by ((26)):

$$\begin{aligned} \int_I u ds &\leq (\tfrac{1}{\sqrt{2b}} W[I]^{\frac{1}{2}})^2 \\ &= \tfrac{1}{2b} W[I]. \end{aligned} \tag{25}$$

Theorem 3.7

Let ϕ_k be a minimizing sequence of Möbius strip immersions. Let $\eta, 0 < \eta < \infty$ be a constant such that

$$\liminf_{k \to \infty} W[\phi_k] \leq \eta.$$

Then there is a subsequence $\{\phi_{k'}\}$ which converges uniformly on (compact subsets of) *R, and the associated frames $\{T_{k'}, N_{k'}, B_{k'}\}$ converge uniformly on R as well.*

Proof. The proof depends on lemma((3.3)). Without loss of generality we may assume that $\forall k, W[\phi_k] \leq \eta$. By the Arzela-Ascoli theorem, it suffices to prove that the sequences of maps $\{\phi_k : R \to R^3\}$, $\{\nu_k : [0, L] \to S^2\}$ and $\{T_k \equiv \dot{\sigma}_k : [0, L] \to S^2\}$ are equi-continuous and bounded.

[6] $\mu \equiv \frac{\tau}{\kappa}$ may be considered as a measure of the planarity of σ [Altschuler]. We remark that even if κ vanishes at a point in I, the first integral on the right hand side of ((24)) is finite since $\frac{u}{\kappa} \equiv \kappa(1+\mu)$. And by ((4)), since we have a sequence of isometric images of a fixed rectangle, $\exists C$ independent of the immersion such that $\mu \leq C$.

Since we posit that the isometries $\{\phi_k\}$ must all contain a fixed point $p \in R^3$ in their image, $\phi_k(R) \subset B_{d/2}(p)$ where $d = diameter(R)$. For all $x, y \in R$,

$$\begin{aligned} dist_{R^3}(\phi_k(x), \phi_k(y)) &\leq dist_{\Sigma_k}(\phi_k(x), \phi_k(y)) \\ &\equiv dist_R(x, y), \end{aligned}$$

since $\forall k$, ϕ_k is an isometry. It follows that there is a subsequence $\phi_{k'}$ which converges uniformly on R.

The frames are automatically bounded as maps $R \to SO(3)$.

For arbitrary $x \in R$, let x^* denote the intersection of the ruling which contains x and the centerline $[0, L] \times \{0\}$ of R. By flatness, the normal is constant along a ruling, so

$$\|\nu_k(x) - \nu_k(y)\| \equiv \|\nu_k(x^*) - \nu_k(y^*)\|$$

but

$$\begin{aligned} \|\nu_k(x^*) - \nu_k(y^*)\| &\leq \int_u^v \|\dot{\nu}_k\| ds \\ &\leq |u - v|^{\frac{1}{2}} (\int_u^v \|\dot{\nu}_k\|^2 ds)^{\frac{1}{2}} \end{aligned}$$

by the Schwarz inequality, where $x^* = (u, 0)$ and $y^* = (v, 0)$ in (u, t) coordinates; letting $I = [u, v]$,

$$\leq \frac{1}{2b} |u - v|^{1/2} (W_k[I])^{1/2} \leq \frac{1}{2b} |u - v|^{1/2} \sqrt{\eta} \tag{26}$$

by Lemma ((3.3)) and the hypothesis, so

$$\|\nu_k(x) - \nu_k(y)\| \leq \frac{1}{2b} \sqrt{\eta} |u - v|^{\frac{1}{2}}.$$

Similarly,

$$\|T_k(x) - T_k(y)\| \equiv \|\dot{\sigma}_k(x) - \dot{\sigma}_k(y)\|$$

$$\leq \| \int_u^v \ddot{\sigma}_k(s) ds \|$$

$$\leq |x-y|^{\frac{1}{2}}\left(\int_u^v \|\ddot{\sigma}_k(s)\|^2 ds\right)^{\frac{1}{2}}$$

$$\equiv \left(\int_u^v \kappa^2\right)^{\frac{1}{2}} |x-y|^{\frac{1}{2}}$$

$$\leq \frac{1}{2b}|x-y|^{\frac{1}{2}}(W[\phi_k])^{\frac{1}{2}}.$$

so

$$\|T_k(x) - T_k(y)\| \leq \frac{1}{2b}|x-y|^{1/2}\sqrt{\eta},$$

again by Lemma ((3.3)).

It follows that $B_{k'} \equiv T_{k'} \wedge N_{k'}$ is uniformly convergent as well.

By taking a common subsequence one obtains the theorem.

4 Regularity.

First, we summarize the relevant arguments in L. Simon's paper.

To show that the limit measure is actually a smooth embedded surface, one must first show that the oscillation of the normals to the minimizing sequence of surfaces is uniformly equibounded. Then by the Ascoli theorem applied to a family of local graphs topologized by Hausdorff distance + integral of the oscillation, one can prove that the limit measure is actually the restriction of two-dimensional Hausdorff measure to a smooth surface. [Simon, p. 203]

After deriving local graph representations of the minimizing surfaces inside balls where the total curvature F is uniformly small in a sense made more precise in Lemma 4.1 below, Simon obtains such an estimate by applying the Poincaré inequality for functions $v \in C_0^1(U, R^n), U \subset R^2$, with mean value 0:

$$\|v\|_2 \leq \left(\frac{|U|}{\omega_2}\right)^{\frac{1}{2}} \|Dv\|_2 \tag{27}$$

where $|U|$ = area of domain U, and ω_2 = area of the unit 2-disc.

Suppose a surface is represented locally as a graph of u. We can estimate the oscillation of the normals over a domain U by integrating the gradient

$Du- < Du >$, where $< Du >$ denotes the average value of Du. We apply the Poincaré inequality to $Du- < Du >$. The right hand side then contains D^2u, which we control by our assumption on locally small $|A|^2$. We need an integral estimate on $|A|^2$ which is good for all sufficiently large k.

Thus, a key component of Simon's argument can be summarized in the following estimate:

Lemma 4.1

Let $B_r(z)$ be a ball of radius $r > 0$, center z. Let z be a point in R^3 (We'll take z in the support M_∞ of the limit measure, minus bad points where curvature decays too slowly as $r \to 0$). *Define*

$$F[r, z, k] = \int_U |A|^2$$

where

A = second fundamental form on k-th surface, and $U = T.M_k \cap B_r(z)$, where $T.M_k$ is the approximating tangent plane to the $k - th$ surface.

Define

$$F[r, z] = \liminf_{k\to\infty} F[r, z, k].$$

Restrict our attention to a small ball $B_r(z)$ such that total curvature $F[r, z, k]$ is small for infinitely many k. For a fixed $\epsilon > 0$ if

$$F[r, z] < \epsilon$$

then for "all" sub-balls, we have the following inequality

$$F[\frac{r_0}{2}, z_0] \leq cF[r_0, z_0] \tag{28}$$

where c is independent of r,z.[7]

[7]We use F to distinguish Simon's integral which differs from our W by a factor of 4.

The proof of this lemma requires local replacement by graphs of biharmonic functions. Since biharmonic functions minimize the integral of $|D^2u|^2$, such replacements don't increase total $|A|^2$ too much.

We turn to our case of a Möbius strip. Biharmonic replacements may not be used in the same way because they minimize $|D^2u|^2$ among *all* competitors, not just Gauss-flat graphs. Among other simplifications, we hope to avoid biharmonic replacement and prove more directly a version of ((2.2)). Better still, we could try to obtain directly, a uniform bound on oscillation of the normals which is independent of k, which would enable us to derive regularity of the limit. Unfortunately, it seems that the inequalities in the proof of Lemma ((3.4)) are not sufficiently strong. We must first exclude a set of bad points where the curvature density accumulates in the sense defined by Simon:

Definition 4.2 *Fix $\epsilon > 0$. Let x be a point in R^3 and let $B_r(x)$ be a ball of radius r centered at x. x is an $\epsilon-$*bad point *with respect to the minimizing sequence M_k if*

$$\liminf_{\rho\downarrow 0}\liminf_{k\to\infty}\int_{B_r(x)\cap M_k}|\mathbf{H}|^2 * 1 \geq \epsilon$$

Colloquially, we will simply call such points x bad points.[8]

Lemma 4.3

For each $\epsilon > 0$, there are a finite number of $\epsilon-$bad points for a minimizing sequence M_k.

Proof. Take a finite number of bad points, say $x_1 ... x_N$. Consider

$$W[M_k \cap \bigcup_{i=1...N} B_r(x_i)].$$

For $r < \frac{1}{2}min\{dist(x_i, x_j)\}$, the balls are disjoint, so

$$W[M_k \cap \bigcup_{i=1...N} B_r(x_i)] = \sum_{i=1}^{N} W[M_k \cap B_r(x_i)]$$

[8] Note that an ϵ-bad point is an η-bad point for all $\eta < \epsilon$. Intuitively, in the neighborhood of such a point, the tangent planes of the surfaces M_k tend to rotate at a rate bounded below as $k \to \infty$.

$$\geq N\epsilon,$$

for sufficiently large k, by the definition of bad point. On the other hand, since we have a minimizing sequence, we may assume that there is an $\eta < \infty$ such that $W[M_k] \leq \eta$ for sufficiently large k, which implies that

$$N \leq \eta/\epsilon$$

Corollary 4.4 *There are no bad points in the interior of M_∞.*

Proof. (Sketch) (We should modify the definition of bad point to ruled $I-$neighborhoods.) Suppose $x \in M_\infty$ is a bad point. Since M_∞ is ruled, x lies on a line segment. Consider approximating sequence $\{x_k, xi_k\} \to \{x, \xi\}$. On each ruling in M_k, the mean curvature H_k is monotone. We claim that if x is in the interior, then every point "upstream" is also a bad point, which contradicts the Lemma.

Whereas Simon works in neighborhoods of points in the target space (R^3), we can work in neighborhoods in the domain rectangle because we have a fixed domain, eg. in (s,v) coordinates, for this family of isometric embeddings. To obtain an estimate like ((22)), given the constraint $W[I] < \epsilon$, we must be able to choose ϵ sufficiently small to eliminate wrinkles for all sufficiently large k, aside from a finite set of bad points in R^3.

More precisely,

Lemma 4.5 *For a fixed flat immersion ϕ, let $W(I)$ be the local integral mean curvature defined by ((15)). Let $\nu(s)$ be the normal to the surface at the point (s,0). Then*

$$osc_I\nu \equiv \sup_{s_0,s_1\in I} |\nu_(s_1) - \nu_(s_0)| \leq C\, W(I). \tag{29}$$

where C is independent of ϕ.

Proof. The proof follows from Lemma ((3.4)) and the proof of Theorem ((3.5)).

We claim that each M_k may be replaced, if necessary, by an isometric Möbius strip with local oscillation of the normal bounded independent of k. This follows from comparing $W(I,\phi)$ with $\int_I \kappa^2\, ds$, the squared curvature of the spine [Langer-Singer, Bryant], and from proving that one can always solve the following problem: Given a constant $C>0$, a curve σ parametrized by arclength s, $\sigma : [0,L] \to R^3$ with $\int_I \kappa^2\, ds < C$, and given a continuous non-negative function[9]

$$\mu : [0,L] \to R^+, \;\; \mu(s) < \frac{\alpha^s}{b}$$

one can find a flat Möbius strip defined by σ and a unit ruling $\xi : [0,L] \to S^2(1)$ such that

$$\begin{aligned} &< T, \xi, \dot{\xi} > \equiv 0 \\ &\forall s, \;\; |\dot{\xi}(s)| \leq \mu(s). \end{aligned} \tag{30}$$

By Lemma 2.1 the first condition guarantees that the surface is flat. This first order ODE is equivalent to

$$< A\xi, \dot{\xi} > = 0 \tag{31}$$

where A is an antisymmetric matrix whose components satisfy

$$< A_{23}, -A_{13}, A_{12} > = \dot{\sigma}. \tag{32}$$

We can find a solution to ((16)) by solving the related ODE

$$\dot{\xi} = \lambda A\xi \wedge \xi \tag{33}$$

where $\lambda(s)$ is some nonzero function, with the boundary conditions $\xi(0) = -\xi(L)$. A general solution exists because the right hand side is lipschitz; in fact $\forall s \in [0,L]$ the norm of the operator $v \to Av \wedge v$ is $\|A\| \equiv 2$. Collecting our argument, we have the following

Lemma 4.6

[9] α is the constant in ((4)). All we need is some universal upper bound on $\dot{\xi}$ independent of the particular immersion; $\mu(s) < C_1 \frac{\sin^2(\theta(s))}{b}$ for some C_1 independent of ϕ would suffice.

Given a $\sigma(s)$ parametrized by arclength on [0,L] one can always construct a flat strip isometric to $[0, L] \times [-b, b]$ with σ as its spine. Moreover, if the curvature satisfies $\kappa_\sigma(0) = 0$, and there exists a solution to ((19)) satisfying $\xi(0) = -\xi(L)$, then the strip is a mobius strip.

Of course, we must show that there indeed exists a solution satisfying the anti-periodicity condition.

Singularities can occur at $\dot{\theta} = 0$ (viz. Lemma 2.2). Requiring that the immersions ϕ_k be analytic would guarantee that the zeroes are isolated. The second condition allows us to choose $|\dot{\theta}|$ as small as necessary to provide uniform bounds on the integrand of $W[\phi]$.

Our next step is to provide an upper estimate on W[I] in terms of some positive power of $\|I\|$, with uniform constants. [10] In view of Lemma 2.2, we observe that

$$\kappa \leq \frac{u}{\kappa} \equiv \kappa(1 + \cot(\theta)^2) \leq C\kappa$$

where C is some absolute constant depending only on the original domain rectangle R.

Next we seek an upper bound on $P(\gamma, b)$. Note that $P(\gamma, b)$ is symmetric in γ, and is convex in γ so that $\sup_{\gamma\in(-a,a)} P(\gamma, b) = P(a, b)$. So, in the interval I,

$$2b \leq P(\gamma, b) \leq P(\sup_I \gamma, b) \tag{35}$$

where

$$\gamma \equiv \frac{\dot{\theta}}{(\sin(\theta)^2)} \leq \frac{1}{\alpha^2}\dot{\theta}$$

so estimating P entails estimating $\dot{\theta} = -\frac{\partial}{\partial s}(\frac{\tau}{\kappa})$. In fact, from equation ((22)), $\|\dot{\xi}\| \equiv |\dot{\theta}| \leq 2|\frac{\lambda}{\sin(\theta)}|$. . .

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[10] Another approach is to expand the log term in (11) with respect to $\gamma \equiv \frac{\dot{\theta}}{\sin^2(\theta)}$. We obtain

$$\begin{aligned} W[I] &= \int_I \frac{\kappa^2}{\gamma \sin^4(\theta)} \ln(\tfrac{1+b\gamma}{1-b\gamma}) ds \\ &= \int_I \frac{\kappa^2}{\gamma \sin^4(\theta)} (2b\gamma + O(\gamma^2)) ds \\ &= 2b \int_I \frac{\kappa^2}{\sin^4(\theta)} (1 + O(\gamma)) ds. \end{aligned} \tag{34}$$

Assuming an upper estimate on $\dot{\theta}$, we conclude that

$$W[I] \leq C \int_I \kappa^2 ds.$$

where C is independent of the particular immersion.

Coupled with the arguments in section 2, we establish that the limit is a smooth ruled surface. That it is a Möbius strip follows from the continuous dependence of the frame on (variations of) the spine.

5 Classical calculus of variations.

To discover some geometric characterization of the canonical Möbius strip, note that ((9)) depends only on the geometry of the spine σ and the global conditions ((3)). It may be useful to consider the variational problem of minimizing $W[I]$ on closed curves $\sigma : [0, L] \to R^3$, subject to constraints

$$\begin{aligned} |\dot{\sigma}| &= 1 \\ < T, \xi > &= \cos(\theta) \\ |\xi| &= 1 \\ < A\xi, \dot{\xi} > &= 0 \end{aligned} \tag{36}$$

where A is the antisymmetric matrix defined in ((18)).

One may also compare this variational problem with the problem of finding elastica of fixed length in space forms. Langer and Singer [Langer] have classified all such elastica and find, in particular, that in R^3 the nonplanar elastica, loops with $\kappa \not\equiv 0$, lie in tori of revolution. Motivated by physical models, one might seek minimizers close to the elastica with three-fold symmetry.

6 Experimental methods for closed surfaces.

We apply the Hamilton-Jacobi formulation of the variational problem [Evans-Spruck]. Take an initial surface represented as the zeroset of $u(x) : \{X : u(X) = 0\}$. We wish to evolve in a normal direction with speed $= f(\text{curvature})$, for all time.

If we consider the family of surfaces represented by

$$\left\{ \begin{array}{c} G_t = \{X : U(X,t) = 0\} \\ U(X,0) = u(X) \end{array} \right. \tag{37}$$

then taking the derivative with respect t:

$$\nabla U . \frac{dX}{dt} + \frac{dU}{dt} = 0 \tag{38}$$

for X in G_t

But since $U \equiv 0$ on G_t, ∇U is normal to G_t.

Also, $\frac{dX}{dt}$ is precisely the motion of the surface G_t, so

$$\frac{dX}{dt} = f(curvature\, of\, G_t)N \tag{39}$$

where N is the unit normal to G_t.

(12) becomes

$$\frac{dU}{dt} = -f|\nabla U|$$

We have applied this scheme to evolving curves in the plane as well as genus 0 and genus 1 surfaces in R^3 by $\mathbf{H}$ and ∇W flows. Evans and Spruck have proven the existence of viscosity solutions to the mean curvature flow past singularities. Unfortunately their methods do not appear to work for the case of the Willmore flow. More recently however, Tatiana Toro [Toro] has used methods of Gerhard Huisken to prove energy estimates which imply short time existence for the Willmore flow. This, coupled with preliminary work by Hsu, Kusner & Sullivan [Hsu], provide hope that an evolution scheme might actually yield minima, given suitable initial data, of course.

References

subset of bibliography for Willmore problem

[1] Ralph Alexander, Lipschitzian mappings and total mean curvature of polyhedral surfaces *Trans. AMS* 288 no. 2 (1985) 661-678.

[2] Steven Altschuler. Singularities of the Curve Shrinking Flow for Space Curves. *J. Diff. Geom.* **34** (1991) 491-514.

[3] Robert Bryant, A duality theorem for Willmore surfaces, *The mathematical Heritage of Hermann Weyl.* (1987) 227-240, Proc. Symp. Pure Math 48 AMS 1988 (John Lee).

[4] ———, Surfaces in conformal geometry, *J. Diff. Geom.* **20** (1984) 23-53.

[5] B.Y. Chen, *Total Mean Curvature and Submanifolds of Finite Type*, World Scientific (1984).

[6] N. Ejiri, A counter example for Weiner's open question, *Indiana Univ. Math. J.* **31** no. 2 (1982), 209-211.

[7] Gilbarg, D.; N. Trudinger. *Elliptic Partial Differential Equations of Second Order,* second edition. SpringerVerlag, Berlin (1983).

[8] Gerhardt Huisken. second variation of W(M) ???

[9] Lucas Hsu, Calculus of variations by the Griffiths formalism, *J. Diff. Geom.*1991.

[10] Osamu Kobayashi, On a conformally invariant functional of the space of Riemannian metrics, *J. Math. Soc. Japan*, **37.3** (1985), 373-389.

[11] ———, A Willmore type problem for $S^2 x S^2$. *Proceedings of Differential Geometry & Differential Equations, Shanghai,* Lecture Note 1255, Springer Verlag (1985), 67-72.

[12] Wolfgang Kuhnel, U. Pinkall, On total mean curvatures. *Quart. J. Math. Oxford (2)* **37** (1986), 437-447. *(There is no lower estimate of W by a sum of betti numbers analogous to total absolute curvature.)*

[13] Kusner, Robert, Comparison surfaces for the Willmore problem. preprint. University of Cal. Berkeley. (1987)

[14] ———, Conformal geometry and complete minimal surfaces. *Bull. A.M.S. (new series)* **17.2** (1987), 291-295.

[15] ————, Global geometry of extremal surfaces in three-space.. Thesis, UC Berkeley (1987).

[16] Joel Langer, David A. Singer. The total squared curvature of closed curves. *J. Diff. Geom.* **20** (1984), 1-22.

[17] —————, Curves in the hyperbolic plane and mean curvature of tori in 3-space. Bull. London Math.Soc. 16 (1984) .

[18] Li, Peter; S.T. Yau, A new conformal invariant and its applications to the Willmore conjecture and the first eigenvalue of compact surfaces. Invent. Math.**69** (1982) 269-291.

[19] S. Montiel, A. Ros, Minimal immersions of surfaces by the first eigenvalue and conformal area. *Invent. Math.* **83** (1986), 153-156. *(shows that there are flat tori a+bi with* $b > 1$ *with small conformal area)*

[20] Frank Morgan, *Geometric Measure Theory: A Beginner's Guide,* Academic Press (1988).

[21] M. Onder, R.W. Tucker, Membrane interactions and total mean curvature. *Phys Letters B* bf 202 no. 4 (?), 501-4.

[22] Palmer, Bennett, personal communications.

[23] U. Pinkall, Hopf Ttori in S^3, Max-Planck Inst.-fur-Mat. Gottfried-Claren- Strasse 26; D-5300 Bonn 3. Preprint (Jan. 1985).

[24] K. Shiohama, R. Takagi, A characterization of a standard torus in E^3, *J. Diff. Geom.* **4** (1970), 477-485. *proves compact connected surface, genus ¿ 0, with one constant principal curvature must have* $W \geq 0$ *with inf achieved by Clifford.*

[25] Simon, Leon, Existence of Willmore surfaces. Miniconference on Partial Differential Equations. Proc. Centre Math. Analysis, Australian Nat. Univ. (1982) 187-216.

[26] Leon Simon, *Lectures on Geometric Measure Theory*, Proc. Centre for Math. Analysis, Australian Nat. Univ. **3** (1983).

[27] A. El Soufi, S. Ilias. Immersions minimales, premiere valeur propre du Laplacien et volume conforme. *Preprint L'Institut Fourier, laboratoire de Mathematiques*, Grenoble France, **42** (1985).

[28] Spivak, M, *A Comprehensive Introduction to Differential Geometry III, IV, V*, Publish or Perish (1975).

[29] Weiner, Joel L, On a problem of Chen, Willmore, et. al. Indiana U. M. J. **27** no. 1 (1978) 19-35.

[30] ———, On an Inequality of P. Wintgen for the integral of the square of the mean curvature. preprint (1980). *[for surfaces in E^4, obtains a lower bound on t in terms of genus, self-intersection and inf of energies of smooth maps $M \to S^2$, given induced conformal structure on M.]*

[31] T.J. Willmore, Note on embedded surfaces, *An. Stiint. Univ. "al. I. Cuza" Iasi Sect. I a mat.*, **(N.S.) 11** (1965), 493-496.

[32] ———, *Total Curvature in Riemannian Geometry* Wiley & Sons (1982).